Data provided: Formula sheet

The University Of Sheffield.

MAS248

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

MATHEMATICS III (CHEMICAL) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Find and classify the stationary points of the function

$$f(x, y) = x^4 + y^4 - 4a^2xy,$$

where a is a constant. ***(9 marks)***

(ii) Write down the iteration formula for the Newton-Raphson method. Starting from $x_0 = 1.0$ use the Newton-Raphson method to find an approximation to a root of the equation

$$x \sin x - 1 = 0,$$

correct to three decimal places. ***(8 marks)***

(iii) If

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t},$$

and

$$u = \frac{x}{2\sqrt{kt}},$$

where k is a constant and $v = v(u)$, show that

$$\frac{d^2v}{du^2} + 2u\frac{dv}{du} = 0.$$

(8 marks)

2 (i) The Dirichlet conditions are sufficient conditions for a real-valued, periodic function $f(x)$ to be equal to the sum of its Fourier series at each point where it is continuous. State two of the Dirichlet conditions. ***(2 marks)***

Explain why the following functions do not satisfy the Dirichlet conditions:

(a) A periodic function, $g(x)$, of period 4π, defined by

$$g(x) = \frac{1}{16 - x^2} \quad \text{for} \quad -2\pi \leq x < 2\pi.$$

(b) A periodic function, $h(x)$, of period 2π, defined by

$$h(x) = \sin\left(\frac{1}{x-1}\right) \quad \text{for} \quad -\pi \leq x < \pi.$$

(4 marks)

(ii) Write down the definition of an odd function and the definition of an even function.

State whether each of the following functions is odd, even or neither even nor odd:

(a) $e^{-|x|}$,

(b) $x \cosh x$,

(c) $x^2 \sin x$,

(d) e^{-4x},

(e) $x\,|x| - x^3$. ***(6 marks)***

(iii) A periodic function, $f(x)$, of period 2π is defined by

$$f(x) = |x| \quad \text{for} \quad -\pi \leq x < \pi.$$

Supposing that $f(x)$ has a convergent trigonometric Fourier series, show that

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right).$$

(10 marks)

Hence deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

(3 marks)

3 (i) The vector field, $\mathbf{A}$, is given by

$$\mathbf{A} = (e^{xy}, 2x - y, y \sin x).$$

(a) Calculate $\nabla \cdot \mathbf{A}$. ***(2 marks)***

(b) Calculate $\nabla \times \mathbf{A}$. ***(4 marks)***

(c) Evaluate $\nabla^2 \mathbf{A}$ at the point with co-ordinates $(\frac{\pi}{2}, 0, 0)$. ***(4 marks)***

(ii) A vector field, $\mathbf{F}$, is given by

$$\mathbf{F} = (3x^2 + yz, z(x-1), y(x-1)).$$

Find a scalar potential, ϕ, for the vector field $\mathbf{F} = \nabla\phi$. ***(9 marks)***

(iii) Find the gradients of the scalar fields:

(a) $\phi = x^3 + y^3 + z^3$, ***(2 marks)***

(b) $\psi = \mathbf{r} \cdot \nabla(x + y + z)$, ***(4 marks)***
where $\mathbf{r} = (x, y, z)$.

4 (i) Show that the function $u(x, y) = f(y + 2x) + xg(y + 2x)$, where f and g are two arbitrary twice differentiable functions, satisfies the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0.$$ ***(8 marks)***

(ii) Use D'Alembert's method to find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

that satisfies the conditions

$$u(x, 0) = \sin x,$$

$$\frac{\partial u}{\partial t}(x, 0) = x^2,$$

where c denotes a constant phase speed. Simplify your answer as much as possible. ***(17 marks)***

End of Question Paper

Formula Sheet

Fourier Series

Suppose that $f(x)$ is defined on the interval $-L \leq x \leq L$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \qquad n = 0, 1, 2, ...,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, 3, ...$$

On the interval $0 \leq x \leq L$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \qquad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

Gradient of a Scalar Field

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla \phi = \text{grad } \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).$$

Chain Rule

1 If $z = f(x, y)$, where $x = x(t)$, $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

2 If $z = f(x, y)$, where $x = x(u, v)$, $y = y(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

3 If $z = f(u, v)$, where $u = u(x, y)$, $v = v(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Maxima and Minima

1 The function $f(x, y)$ has a stationary point at (x_0, y_0) if

$$f_x = f_y = 0 \quad \text{at} \quad (x_0, y_0).$$

2 At (x_0, y_0), the function $f(x, y)$ has:

(i) a minimum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} > 0 \quad \text{at} \quad (x_0, y_0),$$

(ii) a maximum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} < 0 \quad \text{at} \quad (x_0, y_0),$$

(iii) a saddle point if

$$f_{xx}f_{yy} - f_{xy}^2 < 0 \quad \text{at} \quad (x_0, y_0).$$