Data provided: Formula Sheet

MAS250

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

Mathematics II (Materials) **2 hours**

Marks will be awarded for answers to all questions in Section A, and for your best ***THREE*** *answers to questions in Section B. Section A carries 40 marks, and the marks awarded to each question or section of question are shown in italics.*

Section A

A1 Find the general solution of the equation

$$x^2\frac{dy}{dx}-y=\frac{e^{-1/x}}{x}$$ ***(7 marks)***

A2 Find the particular solution of the equation

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=2e^{-3x}$$

which satisfies $\dfrac{dy}{dx}=0$ and $y=1$ when $x=0$. ***(13 marks)***

A3 The pressure (p), volume (V) and temperature (T) of a container full of gas are related by the ideal gas law

$$pV=nRT,$$

where n measures the number of gas particles and R is the universal gas constant.

The pressure is initially 10^5 Nm^{-2}. Using the chain rule, estimate the pressure if the volume decreases by 4% and the temperature increases by 3%. ***(7 marks)***

A4 Two quantities x and y have means 2.36 and 73.4 respectively, standard deviations 3.18 and 2.19 respectively, and covariance -6.87.

(a) Calculate the correlation coefficient between x and y, correct to 3 significant figures. ***(2 marks)***

(b) It is assumed that x and y satisfy the linear relationship

$$y = a + b(x - \bar{x}), \qquad (*)$$

where $\bar{x}$ is the mean of x.

Calculate the least squares estimates of a and b, correct to 3 significant figures. State, giving a reason, whether you expect $(*)$ to give a good model. ***(4 marks)***

A5 Find a vector normal to the surface $\phi = 5$ at the point A with coordinates $(1, 2, 3)$, where

$$\phi = x^2 \cos \pi z + xyz - z \sin \pi y.$$ ***(5 marks)***

Find also the directional derivative of ϕ at A, in the direction $\mathbf{d} = (2, 0, -1)$. ***(2 marks)***

Section B

B1 (a) For $x > 0$, find the particular solution of the equation

$$x\frac{dy}{dx} = y + \frac{x^3}{y^2}$$

which satisfies $y = 3$ when $x = 1$. ***(11 marks)***

(b) Find the general solution of the equation

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = x.$$ ***(9 marks)***

B2 (a) Let $R = (x^2 + y^2)^{1/2}$, and define a scalar field ϕ by

$$\phi = \ln R.$$

Find $\nabla\phi$, and show that

$$\nabla^2\phi = 0.$$ ***(12 marks)***

(b) A vector field $\mathbf{u}$ is given by

$$\mathbf{u} = (x + y \sin z, y + e^z, xz - \cosh y).$$

Verify that

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0,$$

and find $\nabla \cdot \mathbf{u}$. ***(8 marks)***

B3 A function $f(x) = x^2$ is defined on the interval $0 \leqslant x \leqslant 1$.

(a) Show that $f(x)$ can be represented by the Fourier series

$$\frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x).$$

[You may use the fact that $\cos(n\pi) = (-1)^n$.] ***(15 marks)***

(b) Use the result of part (a) to find

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

(5 marks)

B4 The function $\phi(x, y)$ satisfies Laplace's equation in two dimensions, i.e.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

in the square region $0 < x < 1, \quad 0 < y < 1$. The boundary conditions are

$$\begin{aligned} \phi(0, y) &= 0 \\ \phi(x, 0) &= 0 \\ \phi(x, 1) &= 0 \\ \phi(1, y) &= 1. \end{aligned}$$

If $\phi(x, y) = X(x)Y(y)$ for some functions X and Y, show that

$$\frac{X''}{X} = -\frac{Y''}{Y} = \alpha,$$

where α must be a constant. ***(3 marks)***

Find the values of $X(0)$, $Y(0)$ and $Y(1)$. ***(3 marks)***

Given that $\alpha > 0$, deduce that

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n \sinh n\pi x \sin n\pi y$$

for some constants A_n. ***(8 marks)***

Show that

$$A_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{n\pi \sinh n\pi} & n \text{ odd.} \end{cases}$$

(6 marks)

End of Question Paper

FORMULA SHEET

Trigonometry

$1+\tan^2\theta=\sec^2\theta$

$1+\cot^2\theta=\mathrm{cosec}^2\theta$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\cos(A-B)=\cos A\cos B+\sin A\sin B$

$\sin(A+B)=\sin A\cos B+\cos A\sin B$

$\sin(A-B)=\sin A\cos B-\cos A\sin B$

$\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

$\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$\sin 2\theta=2\sin\theta\cos\theta$

$\cos 2\theta=2\cos^2\theta-1=1-2\sin^2\theta$

$a\cos\theta+b\sin\theta=R\cos(\theta-\alpha)$, where $R=\sqrt{(a^2+b^2)}$, $\cos\alpha=a/R$ and $\sin\alpha=b/R$

Hyperbolic Functions

$\sinh x=\dfrac{1}{2}(e^x-e^{-x})$

$\cosh x=\dfrac{1}{2}(e^x+e^{-x})$

$\cosh^2 x-\sinh^2 x=1$

$\mathrm{sech}^2 x+\tanh^2 x=1$

$2\sinh x\cosh x=\sinh 2x$

$\cosh 2x=2\cosh^2 x-1=2\sinh^2 x+1$

$\sinh^{-1}x=\ln\left[x+\sqrt{(1+x^2)}\right],\quad \text{all } x$

$\cosh^{-1}x=\ln\left[x+\sqrt{(x^2-1)}\right],\quad x\geqslant 1$

$\tanh^{-1}x=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right),\quad |\,x\,|<1$

$\coth^{-1}x=\dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right),\quad |\,x\,|>1$

Differentiation and Integration

Function	**Derivative**
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\tan x$	$\sec^2 x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\mathrm{cosec} x$	$-\mathrm{cosec} x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\mathrm{sech}^2 x$
$\coth x$	$-\mathrm{cosech}^2 x$
$\mathrm{sech} x$	$-\mathrm{sech} x \tanh x$
$\mathrm{cosech} x$	$-\mathrm{cosech} x \coth x$
$\sin^{-1} x$	$\frac{1}{\sqrt{(1-x^2)}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{(1-x^2)}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{(x^2+1)}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{(x^2-1)}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}, \quad \mid x \mid < 1$
$\coth^{-1} x$	$-\frac{1}{x^2-1}, \quad \mid x \mid > 1$

Function	**Integral**
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$

Differentiation and Integration Formulae

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\int_a^b uvdx = [u \times (\text{integral of } v)]_a^b - \int_a^b \frac{du}{dx} \times (\text{integral of } v)dx$$

Partial Differentiation

Chain Rule

1. Suppose that $z = f(x, y)$ and that x and y are functions of t, i.e., $x = x(t)$, $y = y(t)$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

2. Suppose that $z = f(x, y)$ and that x and y are functions of the variables r and s, i.e., $x = x(r, s)$, $y = y(r, s)$. Then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

First-Order Differential Equations

1. **Direct Integration**

$$\frac{dy}{dx} = f(x)$$

$$y = \int f(x)dx + C$$

2. **Separation of Variables**

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

3. **Homogeneous Equations**

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

make the substitution $y = zx$ to give

$$z + x\frac{dz}{dx} = f(z)$$

4. **Linear Equations**

$$\frac{dy}{dx} + P(x)y = Q(x)$$

multiply both sides by the integrating factor $e^{\int P(x)dx}$ to give

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}$$

The Second-Order Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b, and c are constants.

General solution is

$$y = \text{Complementary Function} + \text{Particular Integral}$$

The solution, y_c, is given by

(i) $y_c = Ae^{m_1 x} + Be^{m_2 x}$, if m_1 and m_2 real and different,

(ii) $y_c = e^{mx}(A + Bx)$, if m_1 and m_2 real and equal $(m_1 = m_2 = m)$,

(iii) $y_c = e^{px}(A\cos qx + B\sin qx)$, if m_1 and m_2 are complex $(m_1 = p + iq, m_2 = p - iq)$, where m_1 and m_2 are the roots of the *auxiliary equation*

$$am^2 + bm + c = 0$$

Particular Integral, y_p

$f(x) = Ax^2 + Bx + C$ $\qquad y_p = ax^2 + bx + c$

$f(x) = Ae^{kx}$ $\qquad y_p = ae^{kx}$
when k is not one of the roots of the auxiliary equation

$f(x) = Ae^{kx}$ $\qquad y_p = axe^{kx}$
when k is one of the roots of the auxiliary equation

$f(x) = A\cos mx + B\sin mx$ $\qquad y_p = a\cos mx + b\sin mx$
when $\sin mx$ or $\cos mx$ is not part of the complementary function

$f(x) = A\cos mx + B\sin mx$ $\qquad y_p = x(a\cos mx + b\sin mx)$
when $\sin mx$ or $\cos mx$ is part of the complementary function

Fourier Series

Suppose that $f(x)$ is defined on the interval $-l \leqslant x \leqslant l$. The Fourier series for $f(x)$ is given by

$$f(x) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} \;+\; b_n \sin\frac{n\pi x}{l}\right) \;,$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x \;, \quad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \;, \quad n = 0, 1, 2, \ldots.$$

On the interval $0 \leqslant x \leqslant l$ the Fourier cosine series for $f(x)$ is

$$f(x) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \;, \quad a_n \;=\; \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

and the Fourier sine series is

$$f(x) \;=\; \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \;, \quad b_n \;=\; \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \;.$$

Vector Calculus

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla\phi \;=\; \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \;.$$

The divergence of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The curl of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

The Laplacian ∇^2 is given by

$$\nabla^2 \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Statistics

For data values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Means $\quad \bar{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i \quad$ etc.

Variances $\quad s_x^2 = \dfrac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \dfrac{1}{n}\sum_{i=1}^{n}(x_i^2) - \bar{x}^2 \quad$ etc.

s_x is standard deviation

Covariance $\quad \mathrm{cov}(x, y) = \dfrac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \dfrac{1}{n}\sum_{i=1}^{n}(x_i y_i) - \bar{x}\bar{y}$

Correlation coefficient $\quad r = \dfrac{\mathrm{cov}(x, y)}{s_x s_y}$

Linear regression by least squares

The least squares fit to the linear relationship

$$y = a + b(x - \bar{x})$$

is given by

$$a = \bar{y}, \qquad b = \frac{\mathrm{cov}(x, y)}{s_x^2}$$

The corresponding mean square residual is $s_y^2(1 - r^2)$.