Data provided: Formula sheet

MAS253

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014-2015**

Mathematics for Engineering Modelling **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Find the Maclaurin series for $\cos x$ and $\cos^3 x$, where $\cos^3 x$ denotes $(\cos x)^3$, up to and including the term involving x^4. Show that the Maclaurin series suggest that $\cos 3x + 3\cos x = A\cos^3 x$, and find the value of A.

Find the first 3 terms of the Taylor series for $\sin x$ about the point $x = \pi/4$. Use the result to find an approximate value for $\sin(1)$, quoting your result to 3 decimal places. ***(10 marks)***

(ii) Find an expression for the sum of the infinite series

$$\frac{3}{2} + 3x^2 + 6x^4 + 12x^6 + 24x^8 + \ldots \qquad (1)$$

and determine its radius of convergence, R. Hence find the sum of the infinite series

$$\frac{3}{4}x^2 + \frac{3}{4}x^4 + x^6 + \frac{3}{2}x^8 + \ldots \qquad (2)$$

Hint: consider series (1) multiplied by x. ***(8 marks)***

(iii) Use l'Hôpital's rule to evaluate

(a) $$\lim_{x\to 0} \frac{\sinh x - \sin x}{x^3},$$

(b) $$\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right).$$

(7 marks)

2 Let $f(x) = \cos 2x$ be defined on the range $0 \le x \le \pi$, and let $F(x)$ denote its Fourier *sin* series.

(i) Sketch $F(x)$ for the range $-3\pi < x < 3\pi$.

State Fourier's Theorem (Dirichlet's Theorem for the case of a Fourier series), and hence determine the value of $F(\pi)$.

(8 marks)

(ii) Show that

$$F(x) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{2m+1}{(2m+1)^2 - 4} \sin(2m+1)x\,.$$

(14 marks)

(iii) Deduce that

$$\frac{\pi}{4} = \frac{1}{3} + \frac{3}{5} - \frac{5}{21} + \frac{7}{45} - \ldots$$

(3 marks)

3 (i) (a) Calculate *by integration* the Laplace transform of $f(t) = e^{3t}t^2$, and give the condition on s for the result to be valid.

(b) State the filtering property of the Dirac delta function. Hence calculate the Laplace transform of $f(t) = \delta(t - \pi/3)\sin t$ by integration.

(7 marks)

(ii) With the aid of the Table of Laplace transforms, find the inverse Laplace transforms of

$$\text{(a)}\ \frac{s-5}{s^2+2s+5}\,, \quad \text{(b)}\ \frac{1}{(s^2+1)(s-3)}\,,$$

and (c) the Laplace transform of

$$u(t-4)\,t^2\,,$$

where $u(t)$ is the unit Heaviside function. ***(8 marks)***

(iii) The displacement x of a mass attached to the end of a spring follows the governing equation

$$4\frac{\mathrm{d}^2x}{\mathrm{d}\,t^2} + 100\,x = f(t)\,,$$

where $f(t)$ is a force applied to the mass. The mass is initially at $x = 2$ and at rest, when it is released at time $t = 0$. At time $t = t_1 = 3\pi/10$ the mass is subject to an impulsive force; the impulse is of magnitude 50.

Use the method of Laplace transforms to find the displacement $x(t)$.

Simplify your expression for $t \le t_1$ and $t > t_1$, and sketch your result.

(10 marks)

4 The equilibrium temperature $T(x,y)$ in an infinitely long metal sheet of width l satisfies

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0\,,$$

where $a \le x < \infty$ and $0 \le y \le l$.

(i) If $T \to 0$ as $x \to \infty$, using the method of separation of variables show that solutions are of the form

$$T(x,y) = \mathrm{e}^{-\lambda x}(A\cos\lambda y + B\sin\lambda y)\,,$$

where λ is a real positive constant. ***(12 marks)***

(ii) Find the general solution that also satisfies the boundary conditions $T = 0$ along the edges $y = 0$ and $y = l$.

(5 marks)

(iii) If the temperature along the edge $x = a$ is 1, find the complete solution for this case, i.e. calculate the coefficients in the general solution of part (ii). ***(8 marks)***

5 (i) Evaluate the integral

$$\int_0^{\sqrt{\pi/2}} \int_0^x x^2 \sin xy \;\mathrm{d}y\,\mathrm{d}x\,.$$

(6 marks)

(ii) By changing the order of integration, evaluate

$$\int_0^1 \int_{x^2-1}^0 x\,\mathrm{e}^{\,y^2+2y}\;\mathrm{d}y\,\mathrm{d}x\,.$$

(8 marks)

(iii) A region R is given by the limits $x^2 + y^2 \le 4$, $y \ge x$. Using a change of coordinates, evaluate the integral

$$\iint_R \frac{(x^2-y^2)\,y}{(x^2+y^2)^2\,(x^2+y^2-9)}\;\mathrm{d}x\,\mathrm{d}y\,.$$

(11 marks)

End of Question Paper