The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2014–2015

Continuum Mechanics — 2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Write down in full the following expressions:

$$\text{(a)}\ \delta_{ij}x_iy_j \quad \text{(b)}\ t_i = T_{ij}n_j \quad \text{(c)}\ B_{ij}C_{ji} \quad \text{(d)}\ \varepsilon_{ijk}x_iB_{jk}.$$

(11 marks)

(ii) The components of tensor $\mathbf{T}$ in coordinates x_1, x_2, x_3 are given by

$$\hat{\mathbf{T}} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

New Cartesian coordinates, x_1', x_2', x_3', are obtained from the old ones, x_1, x_2, x_3, by a rotation about the x_2-axis through an angle θ. The corresponding transformation matrix is given by

$$\hat{\mathbf{A}} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

(a) Show that $\hat{\mathbf{A}}$ is a proper orthogonal matrix. ***(5 marks)***

(b) Find the matrix $\hat{\mathbf{T}}'$ of components of tensor $\mathbf{T}$ in coordinates x_1', x_2', x_3'. (You can use without proof the relation $\hat{\mathbf{T}}' = \hat{\mathbf{A}}\hat{\mathbf{T}}\hat{\mathbf{A}}^T$, and trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$). ***(6 marks)***

(c) You are given that $\hat{\mathbf{T}}'$ is a diagonal matrix, and the angle θ is acute. Determine θ. ***(3 marks)***

2 (i) Give the definition of a streamline. Show that, in Cartesian coordinates x, y, z, the parametric system of equations for streamlines can be written in the form

$$\frac{dx}{d\lambda} = v_1(\mathbf{x}, t), \quad \frac{dy}{d\lambda} = v_2(\mathbf{x}, t), \quad \frac{dz}{d\lambda} = v_3(\mathbf{x}, t),$$

where $\mathbf{v} = (v_x, v_y, v_z)$ is the velocity and $\mathbf{x} = (x, y, z)$. ***(12 marks)***

(ii) In Cartesian coordinates x, y, where x is the horizontal and y the vertical coordinate, a small-amplitude propagating surface wave on water of depth h is given by velocity

$$v_x = v_0 \cos(kx - \omega t) \cosh[k(y + h)],$$
$$v_y = v_0 \sin(kx - \omega t) \sinh[k(y + h)],$$

where v_0, ω and k are constants. Find the equation of streamlines written in the form $F(x, y) = \text{const}$. ***(13 marks)***

3 (i) Write down the expression for the surface traction, $\boldsymbol{t}$, in terms of the stress tensor, $\mathbf{T}$, and the unit normal to the surface, $\boldsymbol{n}$. Express it both in vector and coordinate form. ***(3 marks)***

(ii) The components of the stress tensor in a viscous fluid is given in Cartesian coordinates by

$$T_{ij} = -p\delta_{ij} + \lambda \frac{\partial v_i}{\partial x_j} + \mu \frac{\partial v_j}{\partial x_i}, \qquad (*)$$

where p, λ and μ are constant, and $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity. Use the condition that the stress tensor is symmetric, $T_{ij} = T_{ji}$, to show that $\lambda = \mu$. ***(5 marks)***

(iii) The plane Poiseuille flow is the flow of a viscous fluid between two rigid boundaries defined in Cartesian coordinates by the equations $x_3 = a$ and $x_3 = a + h$, where a and h are constant and $h > 0$. You are given that the velocity of the plane Poiseuille flow is defined by

$$v_1 = V + \beta(x_3 - \gamma)^2, \quad v_2 = 0, \quad v_3 = 0,$$

where V, β and γ are constant and $V > 0$. Use the condition that the velocity is equal to zero at the rigid boundaries to express β and γ in terms of a, h and V. Use these results to write the expression for v_1. ***(8 marks)***

(iv) You are given that the stress tensor in the plane Poiseuille flow is defined by equation $(*)$ with $\lambda = \mu$. Use the result of part (iii) to calculate the components of the force acting per unit area of the lower boundary. Give your answer in terms of p, μ, h and V. ***(9 marks)***

4 (i) Write down the equation of mass conservation in the Eulerian description. Show that, for an incompressible fluid, this equation reduces to $\nabla \cdot \boldsymbol{v} = 0$. ***(3 marks)***

(ii) The fluid motion is called potential when $\boldsymbol{v} = \nabla\varphi$, where φ is the velocity potential. Show that, in the case of incompressible flow, the potential satisfies the Laplace equation $\nabla^2\varphi = 0$. ***(2 marks)***

(iii) Consider a potential flow of an *ideal* incompressible fluid near a rigid cylinder. The cylinder surface is defined by the equation $x^2 + y^2 = a^2$ in Cartesian coordinates x, y, z. You are given that the flow is planar, i.e. its velocity is parallel to the xy-plane and independent of z. You are also given that the flow velocity is in the negative x-direction far from the cylinder, and its magnitude is V.

(a) Introduce cylindrical coordinates r, ϕ, z with the z-axis coinciding with the cylinder axis and the angle ϕ counted counter-clockwise from the x-axis. Then write down the boundary condition for the velocity at the cylinder surface in terms of φ. ***(4 marks)***

(b) Find the expression for the velocity potential φ in cylindrical coordinates r, ϕ, z. $\Big($You can use without proof the formula

$$\nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\phi^2}\Big).$$

Hint: Look for the solution in the form $\varphi = R(r)\cos\phi$, where $R(r)$ is the function to be determined. ***(16 marks)***

5 (i) You are given that, in equilibrium, the stress tensor $\mathbf{T}$ satisfies the equation written in Cartesian coordinates x_1, x_2, x_3,

$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = 0, \qquad (*)$$

where T_{ij} are the components of the stress tensor $\mathbf{T}$, ρ is the density, and b_i are the components of the body force $\mathbf{b}$. You are also given that, in linear elasticity, the Cartesian components of the stress tensor are given by

$$T_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \qquad (\dagger)$$

where u_i are the Cartesian components of the displacement $\mathbf{u}$, and λ and μ are the Lamé constants. Show that, when $b_i = 0$, in the linear elasticity equation $(*)$ reduces to

$$(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} = 0. \qquad (\ddagger)$$

(8 marks)

(ii) Consider a slab of elastic material. The slab thickness is $h = 20$ cm and it is infinite in two other directions, so that, in Cartesian coordinates x, y, z the elastic material occupies the region $0 \le z \le h$. There are forces applied to the slab boundaries. The density of each of the two forces is equal to $f = 10^8$ N m^{-2}. The force applied at the upper boundary is in the positive x-direction, while the force applied at the lower boundary is in the negative x-direction. You are given that the displacement magnitude both at $z = 0$ and $z = h$ is equal to U, and it is in the positive x-direction at $z = h$ and in the negative x-direction at $z = 0$. You are also given that $\lambda = \mu = 8 \times 10^{10}$ N m^{-2}, where λ and μ are the Lamé constants. Use the equation of the linear elasticity $(\ddagger)$ and the boundary conditions at the slab boundaries to find U.

Hint: Assume that the displacement $\boldsymbol{u}$ has only the x-component that depends on z only. ***(17 marks)***

End of Question Paper