The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

Introduction to Relativity **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) (a) State the two postulates of special relativity. ***(4 marks)***

(b) Define what is meant by an *inertial frame*. Give **two** examples of frames of reference from everyday experience which are approximately inertial. ***(4 marks)***

(ii) Two inertial frames $R : (ct, x)$ and $\tilde{R} : (c\tilde{t}, \tilde{x})$ are related by the *Lorentz transformation*

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}, \qquad \gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$

with $v > 0$.

(a) Show that $c^2\tilde{t}^2 - \tilde{x}^2 = c^2t^2 - x^2$. ***(6 marks)***

(b) Let $O : (0, 0)$ and $A : (0, L)$ be two simultaneous events in R, where $L > 0$. Show that the corresponding events $\tilde{O}$ and $\tilde{A}$ are **not** simultaneous in $\tilde{R}$. In which order do the events occur in $\tilde{R}$? ***(4 marks)***

(c) Draw a *spacetime diagram* showing the coordinate axes for the two frames R and $\tilde{R}$, and the events O and A. ***(3 marks)***

(iii) Briefly describe the phenomenon of *length contraction*. ***(4 marks)***

2 Two inertial frames $R : (ct, x, y, z)$ and $\tilde{R} : (c\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ are related by the transformation

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = L \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \qquad L = \frac{1}{4} \begin{pmatrix} 5 & 0 & -3 & 0 \\ 3 & 0 & -5 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

(i) Write down the 4×4 matrix which is the metric tensor g. ***(2 marks)***

(ii) Show that L is a Lorentz transformation. ***(10 marks)***

(iii) Show that L is proper and orthochronous. ***(6 marks)***

(iv) L may be written as the product $L = L_\rho L_\theta$, where L_ρ is a boost in the x direction and L_θ is a rotation about the z axis. Find L_ρ and L_θ, and hence interpret L geometrically. ***(7 marks)***

3 Consider a particle with four-velocity $U = (c\gamma(u), u\gamma(u), 0, 0)$ moving in an inertial frame R, where

$$\gamma(u) = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$$

(i) (a) Show that $g(U, U) = c^2$. ***(2 marks)***

(b) Is U a spacelike, timelike or null vector? ***(2 marks)***

(ii) Consider an inertial frame $\tilde{R}$ moving relative to R at a uniform velocity v in the positive x-direction.

(a) By calculating $\tilde{U} = LU$, where L is an appropriate Lorentz transformation, show that in $\tilde{R}$ the four-velocity is

$$\tilde{U} = \alpha\,(c, w, 0, 0)$$

where

$$\alpha \equiv \gamma(u)\gamma(v)\left(1 - \frac{uv}{c^2}\right) \qquad \text{and} \qquad w \equiv \frac{u - v}{1 - uv/c^2}.$$

(5 marks)

(b) Show that $\gamma(w) = \alpha$. ***(7 marks)***

(iii) Now consider another inertial frame R' moving relative to R at uniform velocity v in the *positive y-direction.*

(a) By applying an appropriate Lorentz transformation, find U' in R'. ***(6 marks)***

(b) Find the three-velocity of the particle in the frame R'. ***(3 marks)***

4 Consider a particle moving in an inertial frame R with position vector $X(\tau)$ and four-velocity $V(\tau)$ given by

$$V(\tau) = (c\cosh\rho, c\sinh\rho, 0, 0)\,, \qquad \rho = \frac{a\tau}{c},$$

where a is constant. Let $X(\tau = 0) = (0, 0, 0, 0)$ initially.

(i) (a) By integrating $V = \dfrac{dX}{d\tau}$, show that

$$X(\tau) = \frac{c^2}{a}\,(\sinh\rho, \cosh\rho\, - 1, 0, 0)\,.$$

(7 marks)

(b) Find the four-acceleration $A(\tau) = \dfrac{dV}{d\tau}$, and show that

$$g(V, A) = 0, \qquad g(A, A) = -a^2.$$

(7 marks)

(ii) Starting from rest at the event $O : (0, 0)$ in R, the particle accelerates at $a = 1\text{ms}^{-2}$ directly towards a star. The particle reaches the star at the event $S : (cT, d)$ in R, where $d = 10$ light years. Let $c \approx 3 \times 10^8\text{ms}^{-1}$.

(a) Show that $\cosh\rho \approx 2.05$ at event S. Hence deduce the speed of the particle in R at S, expressed as a fraction of the speed of light.

(7 marks)

(b) Find T in years. ***(4 marks)***

5 (i) Define the *rest mass* and *four-momentum* of a particle. ***(3 marks)***

(ii) (a) Three identical particles of rest mass m are in uniform motion in an inertial frame R with identical speeds u. They are moving in the positive x-direction, positive y-direction and negative y-direction, respectively. Show that the total four-momentum P in R is

$$P = m\left(3c\gamma(u), u\gamma(u), 0, 0\right).$$

(4 marks)

(b) The three particles undergo a collision at a single point and fuse together. Assuming conservation of four-momentum, find the speed of the composite particle in R. ***(5 marks)***

(c) Show that the rest mass of the composite particle is

$$m\sqrt{\frac{9c^2 - u^2}{c^2 - u^2}}.$$

(7 marks)

(iii) A single photon has energy $E = 2 \times 10^{-19}$J and zero rest mass.

(a) Use $E^2 = p^2c^2 + m^2c^4$ with $c \approx 3 \times 10^8 \text{ms}^{-1}$ to calculate p, the magnitude of the single photon's momentum. ***(2 marks)***

(b) Suppose 10^{20} such photons are directly reflected by a 'solar sail' of mass 1g which is initially at rest. Estimate the resulting speed of the solar sail. ***(4 marks)***

End of Question Paper