The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2014–15

WAVES 2 hours

Marks will be awarded for your best FOUR *answers. The marks awarded to each question or section of question are shown in italics.*

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1 The one-dimensional wave equation for $\phi(x,t)$ is

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}.$$

(i) *Derive* d'Alembert's general solution for the one-dimensional wave equation on $-\infty < x < \infty$ for $t \geq 0$.

(13 marks)

(ii) Given that $c = 1$ and at $t = 0$

$$\phi(x,0) = \cos 2kx, \qquad \frac{\partial\phi}{\partial t} = 2k\sin 2kx,$$

where k is a constant, find $\phi(x,t)$.

(7 marks)

(iii) Give a physical interpretation of your solution. Further, explain why the solution, subject to the initial conditions in (ii), cannot (or can) be a standing wave.

(5 marks)

2 A uniform finite string of length L and density ρ undergoes small transverse vibrations with displacement $y(x,t)$, where $y_{tt} = c^2 y_{xx}$, and c^2 is a constant.

(i) Given that $y(0,t) = y(L,t) = 0$, *derive* by using the method of separation of variables that the general solution is

$$y(x,t) = \sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right\}\sin\left(\frac{n\pi x}{L}\right),$$

where $\{a_n\}$, $\{b_n\}$ are constants.

(15 marks)

(ii) Find $\{a_n\}$ and $\{b_n\}$ for the case when

$$y(x,0) = A\left[\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{3\pi x}{L}\right)\right]; \qquad y_t(x,0) = 0,$$

where A is constant.

(10 marks)

3 In a compressible and uniform fluid the equilibrium density and pressure are ρ_0 and p_0, respectively. Due to the passage of a compressible perturbation, there are *linear* changes in density ρ and pressure p, and the resulting fluid velocity is $\mathbf{u}(\mathbf{x}, t) = u(\mathbf{x}, t)\mathbf{i} + v(\mathbf{x}, t)\mathbf{j} + w(\mathbf{x}, t)\mathbf{k}$, where u, v and w are small.

(i) Given that the exact equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0,$$

obtain a *linear* approximation to this equation.

(4 marks)

(ii) In the *linear* approximation Newton's Second Law can be given by

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

plus two similar equations. Given that $p \equiv p(\rho)$ only, show that

$$\frac{\partial^2 \rho}{\partial t^2} = c^2\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2}\right),$$

where c^2 is a constant which should be defined.

(8 marks)

(iii) In a particular case $\rho = \rho(r, t)$, where (as usual) $r^2 = x^2 + y^2 + z^2$. Show that

$$\frac{\partial^2 \rho}{\partial t^2} = c^2\left(\frac{\partial^2 \rho}{\partial r^2} + \frac{2}{r}\frac{\partial \rho}{\partial r}\right). \qquad (*)$$

Find the partial differential equation satisfied by $R = r\rho$, and hence write down the general solution of $(*)$. (You may quote d'Alembert's general solution of the one-dimensional wave equation.)

(13 marks)

4 There are progressive surface waves on deep water, propagating in the Ox direction. The unperturbed surface is $z = 0$, where z is measured vertically upwards and the wave height is $\eta(x, t)$. The wavelength is short enough for surface tension to be dominant. The velocity potential ϕ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \qquad (*)$$

and the boundary conditions:

$$\text{(a)}\ \frac{\partial \phi}{\partial z} \to 0 \ \text{ as } \ z \to -\infty\,; \quad \text{(b)}\ \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \ \text{ at } \ z = 0\,;$$

$$\text{(c)}\ \frac{\partial \phi}{\partial t} = \frac{T}{\rho}\frac{\partial^2 \eta}{\partial x^2} \ \text{ at } \ z = 0\,,$$

where T is the (constant) magnitude of the surface tension and ρ is the density of the water. Take

$$\phi = f(z)\sin(kx - \omega t),$$

where k and ω are positive constants.

 Question 4 continued on next page

4 (continued)

(i) Show from $(*)$ and condition (a) that $f(z) = Ae^{kz}$ where A is a constant.

(6 marks)

(ii) Use conditions (b) and (c) to find η and to show that $\rho\omega^2 = Tk^3$.

(8 marks)

(iii) Deduce that the phase velocity c and the group velocity c_g are related by

$$2c_g = 3c.$$

(5 marks)

(iv) Given that $T \approx 7.4 \times 10^{-2}$ kg s^{-2} and $\rho \approx 10^3$ kg m^{-3}, find c for waves with wavelength 5×10^{-3} m.

(6 marks)

5 (i) In a model of traffic flow in the direction of Ox, the density of traffic at time t is $\rho(x,t)$, the speed of traffic of density ρ is $v = v(\rho)$, the flowrate $q(\rho) = \rho v(\rho)$, and $c(\rho) = q'(\rho)$. *Given* that $\rho_t + c\rho_x = 0$, show that $c_t + c\,c_x = 0$. If $\rho(x,0) = f(x)$, deduce that in regions where $c(x,t)$ is continuously differentiable:

$$c = c\{f(\xi)\} = F(\xi) \text{ on straight lines } x = \xi + F(\xi)t.$$

(12 marks)

(ii) Using the method of characteristics solve the equation

$$yz_x + xz_y = xy,$$

given that $z = e^{-y^2}$ on $x = 0$ for $y \geq 0$ and that $z = e^{-x^2}$ on $y = 0$ for $x \geq 0$. $\left[\textit{We use the notation } z_x = \dfrac{\partial z}{\partial x} \textit{ etc.}\right]$

(13 marks)

End of Question Paper