The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014-2015**

Complex Analysis **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Express the quotient

$$\frac{2+11i}{3+4i}$$

in the form $x+iy$. ***(2 marks)***

(ii) Express

$$\frac{(1+i)^9}{(1-\sqrt{3}i)^{11}}$$

in the form $re^{i\theta}$ with $r>0$ and $-\pi<\theta\leq\pi$. ***(4 marks)***

(iii) State, without proof, the triangle inequalities for $|\,z+w|$ and $|\,z-w|$. ***(1 mark)***

(a) Show that, if $|\,z+2+3i\,| \leq 1$, then

$$3 \leq |\,2z+1+2i\,| \leq 7.$$ ***(3 marks)***

(b) Sketch the set $S = \{z\in\mathbb{C} : -3\leq \text{Re}z\leq 3, \text{ and } -3\leq \text{Im}z\leq 0\}$. Show that for all $z\in S$,

$$\left|\frac{\cos z}{e^z}\right| \leq e^3\cosh 3.$$ ***(4 marks)***

(iv) Find all the solutions of the following equation:

$$\cosh z = -3.$$

You should express your answers in the form $x+iy$. ***(4 marks)***

(v) The path γ is the arc of the circle $|\,z+1| = 1$ from 0 to -2 given by $z=-1+e^{it}$ $(0\leq t\leq\pi)$. Evaluate

$$\int_\gamma \text{Re}\, z\, dz\,, \qquad \int_\gamma z\,\sin(z^2)\cos(z^2)\,dz\,.$$ ***(7 marks)***

2 (i) Define what is meant by the following two statements:

(a) A function f is **differentiable at the point** z_0;

(b) A function f is **analytic in a region** D. ***(2 marks)***

Let

$$g(z) = \frac{\sinh(z)}{1+z^5}.$$

Decide where g is analytic giving reasons for your answer. ***(4 marks)***

(ii) State, without proof, the Cauchy-Riemann equations for a differentiable function. ***(1 mark)***

Let $h(z) = \mathrm{Re}z + \mathrm{Im}z$ for all $z \in \mathbb{C}$. Prove that the function h is nowhere differentiable. ***(3 marks)***

(iii) In each of the following cases, determine whether there is a function k analytic on $\mathbb{C}$ with $\mathrm{Re}\,(k(x+iy)) = u(x,y)$, giving reasons for your answers:

$$\begin{aligned} &\text{(c)} \quad u(x,y) &=& \cosh x + \cosh y\,, \\ &\text{(d)} \quad u(x,y) &=& \cosh x \cos y - 2\sinh x \sin y + 1\,. \end{aligned}$$

When k exists, find an explicit expression for $k(z)$ in terms of z and show that you have found all the functions satisfying the conditions. ***(9 marks)***

(iv) The path α is the semi-circle from the point 1 to the point -3, given by $z = -1 + 2e^{it}$ $(0 \le t \le \pi)$. Show that

$$\left| \int_\alpha \frac{e^z \sin z}{\mathrm{Re}(z+4)}\,dz \right| \le 2\pi e \cosh 2.$$ ***(6 marks)***

3 State, without proof, Cauchy's Theorem and Cauchy's Integral Formulae for a function and for its derivatives. Your statement should include conditions under which the results are valid. ***(7 marks)***

Let γ be the square contour with **vertices** $2,\ 1+i,\ 0,\ 1-i$ described in the anti-clockwise direction. Without using the Residue Theorem, evaluate

$$\text{(i)}\quad \int_\gamma \frac{\cosh z}{z-1}\,dz\,, \qquad \text{(ii)}\quad \int_\gamma \frac{2\cos z + \cosh z}{z^2+2}\,dz\,,$$

$$\text{(iii)}\quad \int_\gamma \frac{e^z}{4z^2-1}\,dz\,, \qquad \text{(iv)}\quad \int_\gamma \frac{\sin(\pi z)}{(z-1)^8}\,dz\,.$$

(14 marks)

Let the contour α be the circle $|z-1| = 2$ described in the positive direction. Evaluate

$$\int_\alpha [z^3 + \mathrm{Re}(z+1)]\,dz\,.$$

(4 marks)

4 (i) Define what is meant by the statement that the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ has radius of convergence c. ***(1 mark)***

Find the radius of convergence of the power series

$$\sum_{0}^{\infty}(-1)^n \frac{z^{3n}}{(3+4i)^n} \ .$$ ***(4 marks)***

(ii) Show that if the function f has a zero of order k at α, then $\frac{1}{f}$ has a pole of order k at α. ***(5 marks)***

For each of the following functions, find **all the singularities** in $\mathbb{C}$. Classify these singularities giving reasons for your answers and evaluate the residue at each of them:

(a) $\dfrac{1}{e^{i\pi z}\sinh \pi z}$, ***(4 marks)***

(b) $(z+1)\sin\left(\dfrac{1}{z-1}\right)$, ***(4 marks)***

(c) $\dfrac{\sin^2(\pi z)}{(z+1)^2}$, ***(3 marks)***

(d) $\dfrac{\sin(\pi z)}{(z+1)^6}$. ***(4 marks)***

5 (i) Let γ be the rectangular contour with **vertices** $5{+}4i$, $-5{+}4i$, $-5{-}4i$, $5{-}4i$ described in the anti-clockwise direction. Evaluate

$$\int_\gamma \frac{\sin z}{1+e^z}\,dz\ , \qquad \int_\gamma z\,\exp\left(\frac{1}{(z-2)^2}\right)\,dz\ .$$ ***(11 marks)***

using Cauchy's Residue Theorem

(ii) Let $\alpha > 0$. Prove that

$$\int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2+2x+5}\,dx = \frac{\pi \cos \alpha}{2e^{2\alpha}}.$$ ***(10 marks)***

Hence, or otherwise, evaluate

$$\int_{-\infty}^{\infty} \frac{\cos^2 x}{x^2+2x+5}\,dx$$ ***(4 marks)***

You may assume that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5} = \frac{\pi}{2}\ .$$

End of Question Paper