The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014-15**

Fields **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$ (where $i^2 = -1$):

(a) $J_1 = \{a + bi\sqrt{7} : \ a, b \in \mathbb{Q}\}$, ***(4 marks)***

(b) $J_2 = \{a + bi + ci\sqrt{3} : \ a, b, c \in \mathbb{Q}\}$. ***(3 marks)***

(ii) Let $L = \mathbb{Q}(\sqrt{2}, i\sqrt{3})$.

(a) Show that $L = \mathbb{Q}(2\sqrt{2} + i3\sqrt{3})$. ***(5 marks)***

(b) Express the element $\dfrac{1}{1 + 2\sqrt{2} + i3\sqrt{3}}$ in the form

$$a + b\sqrt{2} + ci\sqrt{3} + di\sqrt{6}$$

for some $a, b, c, d \in \mathbb{Q}$. ***(4 marks)***

(iii) Let p be a prime number. Prove that

$$\phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is an irreducible polynomial in $\mathbb{Q}[x]$. ***(9 marks)***

2 (i) Let $K \subseteq L$ be a field extension. What is meant by saying that an element $b \in L$ is *algebraic* over K? ***(2 marks)***

(ii) Is the element $b = \sqrt[5]{\sqrt{2}+\sqrt{3}}$ algebraic over $\mathbb{Q}$? Justify your response and if the answer is 'yes' then find a polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(b) = 0$. ***(5 marks)***

(iii) Let $K \subseteq L$ be a field extension. Suppose that an element $c \in L$ is algebraic over K.

(a) Give a definition of the *minimal polynomial* $m(x) \in K[x]$ of the element c over K and prove that it is an irreducible polynomial over K. ***(6 marks)***

(b) Suppose that $n = \deg(m(x))$. Prove that the set of elements $1, c, c^2, \ldots, c^{n-1}$ form a basis for the vector space $K(c)$ over the field K. ***(9 marks)***

(c) Find the minimal polynomial $m(x) \in \mathbb{Q}(\sqrt{2})[x]$ of the element $c = \sqrt{2} + 2i\sqrt{3}$ over the field $\mathbb{Q}(\sqrt{2})$. ***(3 marks)***

3 (i) Define the content of a polynomial and find the content of the the polynomial $p = \sum_{n=10}^{100} n!x^n$. ***(4 marks)***

(ii) State Gauss's Lemma. ***(3 marks)***

(iii) Prove Gauss's Lemma. ***(9 marks)***

(iv) Show that the polynomial $f(x) = 3x^2+2015x+2$ is an irreducible polynomial over $\mathbb{Q}$. ***(9 marks)***

4 (i) Let B be a set of at least two points in the plane $\mathbb{R}^2$ and $P \in \mathbb{R}^2$. What is meant by saying that

(a) P is *constructible in one step from* B, ***(2 marks)***

(b) P is *constructible from* B. ***(3 marks)***

(ii) Let $a, b \in \mathbb{R}$ be constructible real numbers. Show that $a - b$ and a/b (provided $b \neq 0$) are also constructible real numbers. ***(5 marks)***

(iii) (a) Give a criterion for the regular n-gon to be constructible. ***(3 marks)***

(b) Which of the following regular n-gons can be constructed (justify your response):

$$n = 6,\ 14,\ 68.$$

(5 marks)

(c) Let $m, n \geq 3$ be natural numbers such that the regular m-gon and the regular n-gon can be constructed. Show that the regular $2^k n$-gon can be constructed for all $k \geq 1$ and that the regular mn-gon can be constructed provided that the numbers m and n are co-prime. ***(7 marks)***

End of Question Paper