The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

Combinatorics **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) State the Binomial Theorem. ***(2 marks)***

(ii) Let k, r, s be positive integers, with $1 \leq k \leq r+s$. Prove that

$$\sum_{j=0}^{r} \binom{r}{j}\binom{s}{k-j} = \binom{r+s}{k}$$

by each of the following methods.

(a) Use the expansion of $(1+x)^{r+s}$. ***(4 marks)***

(b) Consider the number of ways of choosing k items from $r+s$ items, where r items are red and s items are blue. ***(4 marks)***

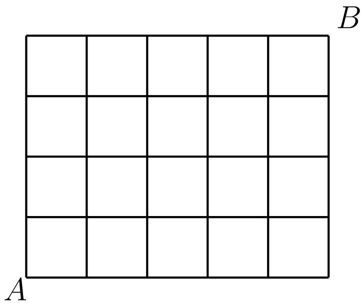
(iii) Consider the equation

$$x_1 + x_2 + \cdots + x_k = n.$$

(a) How many solutions are there of this equation in which each x_i is a non-negative integer? Give a brief reason for your answer. ***(3 marks)***

(b) How many solutions are there of this equation in which each x_i is a positive integer, with $x_i \geq 2$? ***(3 marks)***

(iv) Consider a rectangular grid of roads with k rows and n columns. (The case $k = 4$ and $n = 5$ is pictured below.)



(a) How many shortest routes are there from the bottom left point A to the top right point B along the lines of the grid? ***(2 marks)***

(b) How many of the shortest routes from A to B travel horizontally along part of each of the $k+1$ horizontal lines?

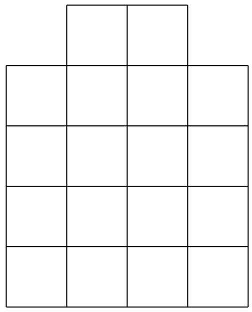
How many of the shortest routes from A to B travel vertically up part of each of the $n+1$ vertical lines?

How many of the shortest routes from A to B travel horizontally along part of each of the $k+1$ horizontal lines **and** up part of each of the $n+1$ vertical lines? ***(7 marks)***

2 (i) Consider a rectangle n squares high and $n-1$ squares wide.

(a) Show that this can be completely covered by non-overlapping dominoes (that is, by pieces which cover exactly two adjacent squares). ***(2 marks)***

(b) Consider the rectangle with the two top corner squares removed. (The case $n=5$ is pictured below.)



Show that this can be completely covered by non-overlapping dominoes if and only if n is odd. ***(4 marks)***

(ii) Suppose that you are given 22 (not necessarily different) integers such that when you multiply them together you get 1. Show that when you add them up it is impossible to get 0. ***(4 marks)***

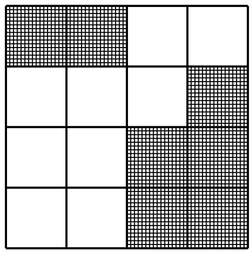
(iii) (a) State the Pigeon-hole Principle. ***(2 marks)***

(b) Given 9 numbers from $\{1, 2, 3, \ldots, 26\}$, show that there are two different subsets each with 5 elements such that the sum of their elements is the same. ***(4 marks)***

(iv) (a) State the Inclusion/Exclusion Principle. ***(3 marks)***

(b) Use the Inclusion/Exclusion Principle to find the number of permutations of the numbers $1, 2, \ldots, 9$ such that at least one odd number is fixed. ***(6 marks)***

3 (i) Calculate the rook polynomial of (the unshaded part of) the board:



(6 marks)

(ii) For each positive integer n, let F_n denote the full $n \times n$ board.

(a) Explain briefly why the rook polynomial of F_n is

$$\sum_{k=0}^{n} \binom{n}{k}^2 k! x^k.$$

(4 marks)

(b) Let $n > 1$ and define B to be the full $n \times n$ board F_n with any one square shaded. Show that

$$r_B(x) = r_{F_n}(x) - x r_{F_{n-1}}(x)$$

and hence find the rook polynomial of B. ***(6 marks)***

(c) Deduce that

$$n! + \sum_{k=1}^{n} (-1)^k (n-k)! \left(\binom{n}{k}^2 k! - \binom{n-1}{k-1}^2 (k-1)! \right) = 0.$$

(3 marks)

(iii) (a) Show that it is possible to have a tournament of n players with all the scores equal if and only if n is odd. ***(3 marks)***

(b) Hence, or otherwise, show that there exists a tournament with $2m$ players, such that half the players score m and the other half score $m - 1$. ***(3 marks)***

4 (i) For what value of x can the following Latin rectangle be extended to a 6×6 Latin square?

$$\begin{pmatrix} 1 & 2 & 6 & 4 \\ 4 & 3 & 5 & 2 \\ 3 & 5 & 2 & 6 \\ 5 & 4 & x & 3 \end{pmatrix}$$

Write down one such extension. ***(8 marks)***

(ii) Prove that there exist at most $n-1$ mutually orthogonal $n \times n$ Latin squares. ***(8 marks)***

(iii) Consider a (v, b, r, k, λ) design, where v is the number of varieties and b is the number of blocks. Explain the meanings of the other parameters r, k and λ. ***(3 marks)***

(iv) Consider all choices of three different numbers from $\{1, 2, 3, 4, 5\}$. Show that these form the blocks of a design. List all the blocks and determine all the parameters of the design. ***(6 marks)***

End of Question Paper