Data provided: list of formulae

The University Of Sheffield.

MAS336

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

Differential Geometry **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

A list of formulae is provided on the last page.

1 Fix $p > q > 0$ and consider the **ellipse** defined by the Cartesian equation

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1. \tag{1}$$

The **eccentricity** of the ellipse is $\epsilon = \sqrt{1 - \frac{q^2}{p^2}}$ and the points $F_1 = (-\epsilon p, 0)$ and $F_2 = (\epsilon p, 0)$ on the x-axis are called the **foci** of the ellipse.

(i) Verify that $\gamma\colon \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (p\cos t, q\sin t)$ is a parametrisation of the ellipse and show that it is regular and smooth. ***(5 marks)***

(ii) Prove that the distance from F_1 to a point $P = \gamma(t)$ on the ellipse is $p(1 + \epsilon\cos t)$, and that the distance from F_2 to $P = \gamma(t)$ is $p(1 - \epsilon\cos t)$ and deduce that the sum of the distances from F_1 and F_2 to any point P on the ellipse does not depend on P. ***(9 marks)***

(iii) Show that for each $t \in \mathbb{R}$, the vectors $\mathbf{n}(t) = (q\cos t, p\sin t)$ and $\dot\gamma(t)$ are orthogonal. ***(3 marks)***

(iv) Compute the curvature function of the parametrised ellipse and find its maxima and minima on $[0, 2\pi]$. ***(8 marks)***

2 Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 + y^2$. We parametrise its graph S by $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^3$, $(u,v) \mapsto (u, v, u^2 + v^2)$.

(i) Is this parametrised surface smooth and regular? ***(4 marks)***

(ii) Compute the first and second fundamental forms of σ. ***(7 marks)***

(iii) Compute the Weingarten matrix and the Gaussian curvature. ***(6 marks)***

(iv) What is the preferred unit normal vector at $P = \sigma(0,0)$? Explain geometrically the meaning of the normal curvature of an arbitrary vector $X \in T_P S$ of length one and compute its value. ***(8 marks)***

3 (i) Parametrise the plane in $\mathbb{R}^3$ through the point $(1,1,0)$ and with directions $(1,0,1)$ and $(0,1,1)$. ***(1 mark)***

(ii) Does your parametrisation preserve lengths? angles? areas? ***(4 marks)***

(iii) Now parametrise the plane in $\mathbb{R}^3$ through the point $(0,0,1)$ and with directions $(1,0,0)$ and $(0,1,0)$. ***(1 mark)***

(iv) Does your parametrisation preserve lengths? angles? areas? ***(4 marks)***

(v) Prove that if a parametrised surface preserves surface areas and angles then it preserves distances. ***(6 marks)***

(vi) Find an example of a parametrised surface that preserves angles, but does not preserve distances and does not preserve surface areas. Find an example of a parametrised surface that preserves surface areas, but does not preserve angles and does not preserve distances. ***(4 marks)***

(vii) Is there a chart of the Earth (say of a sphere of radius 1) that preserves surface areas and angles? ***(5 marks)***

For this question, you can use everything that you have learned on charts of the Earth.

4 We consider here the surface of revolution defined by rotating a circle contained in the xz-plane, of centre $(R, 0, 0)$ and radius r with $R > r > 0$, around the z-axis. This surface is called the **torus**.

(i) Give a parametrisation $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ of the circle contained in the xz-plane, of centre $(R, 0, 0)$ and radius r. ***(2 marks)***

(ii) Show that $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\sigma(t, \theta) = ((R + r\cos t)\cos\theta, (R + r\cos t)\sin\theta, r\sin t)$$

is a parametrisation of the torus. ***(4 marks)***

(iii) Compute the first and second fundamental forms of the torus, as well as its Weingarten matrix and its Gaussian curvature. ***(14 marks)***

(iv) Find the principal curvatures and directions at the point $P = \sigma(0, 0) = (R + r, 0, 0)$ and interpret geometrically your result. ***(5 marks)***

5 (i) (a) Consider the parametrised curve $\gamma_1\colon \mathbb{R} \to \mathbb{R}^2$,

$$\gamma_1(t) = \left(\int_0^t \cos(s^2)\, ds, \int_0^t \sin(s^2)\, ds\right).$$

Compute the arc-length of γ_1 between $\gamma_1(0)$ and $\gamma_1(T)$ for some $T > 0$. Compute the curvature of γ_1. ***(4 marks)***

(b) Find the unique unit-speed parametrised curve $\gamma_2\colon \mathbb{R} \to \mathbb{R}^2$ with $\gamma_2(0) = (1, 0)$, $\dot{\gamma}_2(0) = (1, 0)$ and with curvature $\kappa\colon \mathbb{R} \to \mathbb{R}$, $\kappa(t) = t$. ***(5 marks)***

(c) Find the unique unit-speed parametrised curve $\gamma_3\colon \mathbb{R} \to \mathbb{R}^2$ with $\gamma_3(0) = (5, 8)$, $\dot{\gamma}_3(0) = (0, 1)$ and with curvature $\kappa\colon \mathbb{R} \to \mathbb{R}$, $\kappa(t) = t$, by rotating and translating the previously obtained curve γ_2. Justify carefully your method. ***(7 marks)***

(ii) (a) A compact surface can be triangulated with 2 faces, 3 edges and 1 vertex. Can the Gaussian curvature be strictly positive everywhere on the surface? Justify your answer. ***(5 marks)***

(b) Let S be a surface whose Gaussian curvature is -1 everywhere. Prove that the area of any geodesic triangle on S is less than π. ***(4 marks)***

End of Question Paper

List of Formulae

- The inverse of a 2×2-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with coefficients in $\mathbb{R}$ and $ad - bc \neq 0$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- The cross-product of 2 vectors $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ is

$$v_1 \times v_2 = (y_1 z_2 - z_1 y_2, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1) \in \mathbb{R}^3.$$

- The angle θ between two vectors v_1 and $v_2 \in \mathbb{R}^3$ is given by

$$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\| \, \|v_2\|}.$$

For a curve on $\mathbb{R}^2$ parametrised by $\gamma\colon (\alpha, \beta) \to \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$:

- The arc length from $\gamma(a)$ to $\gamma(b)$, $\alpha < a \leq b < \beta$ is:

$$\int_a^b ||\dot{\gamma}(t)|| \, dt$$

- The curvature of γ at $\gamma(t)$ is

$$\kappa(t) = \frac{\ddot{\gamma}(t) \cdot J(\dot{\gamma}(t))}{\|\dot{\gamma}(t)\|^3} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)^2 + y'(t)^2]^{3/2}},$$

  where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the counterclockwise rotation of angle $\pi/2$.

For a parametrised surface patch $\sigma\colon U \to \mathbb{R}^3$, with U an open set in $\mathbb{R}^2$:

- The first fundamental form is given by

$$\mathrm{I}_{(u,v)} = \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix}$$

  for all $(u, v) \in \mathbb{R}^2$, with $E = \sigma_u \cdot \sigma_u$, $F = \sigma_u \cdot \sigma_v$ and $G = \sigma_v \cdot \sigma_v$.

- The preferred unit normal vector along σ is given by $\mathbf{n}\colon U \to \mathbb{R}^3$,

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||}.$$

- The second fundamental form of σ at $(u,v) \in U$ is

$$\mathrm{II}_{(u,v)} = \begin{pmatrix} L(u,v) & M(u,v) \\ M(u,v) & N(u,v) \end{pmatrix}$$

  where $L = \sigma_{uu} \cdot \mathbf{n}$, $M = \sigma_{uv} \cdot \mathbf{n}$ and $N = \sigma_{vv} \cdot \mathbf{n}$.

- The Weingarten matrix of σ is

$$W = \mathrm{I}^{-1}\,\mathrm{II}\,.$$

- The Gaussian curvature is

$$K = \det W.$$

Gauss-Bonnet formula for a geodesic triangle Δ on a surface:

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + \iint_\Delta K dA$$

where $\alpha_1, \alpha_2, \alpha_3$ are the interior angles of Δ.

Gauss-Bonnet formula for a compact surface S:

$$\iint_S K dA = 2\pi\chi(S).$$