Data provided: Formula sheet

MAS381

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2014–15

Mathermatics III (Electrical) — 2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) The vector field $\mathbf{F}$ is given by

$$\mathbf{F} = (x^2 - 2y)\mathbf{i} + (2x - z)\mathbf{j} + (y + z^2)\mathbf{k}.$$

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the unit circle $x^2 + y^2 = 1$, $z = 0$, traversed counterclockwise, starting and finishing at the point $(1, 0, 0)$. ***(10 marks)***

(ii) Find the Laurent series expansion of

$$f(z) = \frac{1}{2j - (1 + 2j)z + z^2},$$

in the region $1 < |z| < 2$. ***(15 marks)***

2 (i) Find all the poles of the function

$$f(z) = \frac{(z-1)^3}{z^2(z^2 - 2z + 5)},$$

and plot them on an Argand diagram. Hence, evaluate the integral $\oint_C f(z)dz$, writting your solution in the form $a + jb$, where a and b are real, and

(a) C is the circle $|z| = 3$

(b) C is the circle $|z - 4| = 1$. ***(15 marks)***

(ii) By constructing a suitable contour integral in the complex plane, use the method of residues to evaluate the real integral

$$I = \int_{-\infty}^{\infty} \frac{1}{5 + 2x + x^2} dx.$$

(10 marks)

3 (i) Assuming that $z = x + jy$ is a complex number, sketch the regions in the z-plane corresponding to (i) $x \leq -2$, (ii) $y \geq x - 3$, (iii) $|z| \geq 4$, and (iv) $|z - 2 + j| \leq 1$. ***(8 marks)***

(ii) Let a and b be two complex numbers. The mapping $w = az + b$ maps the point $z = -1$ to the point $w = 1 + j$ and the point $z = -1 + j$ to the point $w = 2 + 3j$.

(a) Show that $a = 2 - j$ and $b = 3$.

(b) Using the above values of a and b determine the real-valued functions $u(x, y)$ and $v(x, y)$ given that $w = u + jv$.

(c) Find the image in the w-plane of the region $|z| \leq 5$ in the z-plane under this mapping. ***(12 marks)***

(iii) Determine whether the image of the straight line $z = t + 3 + j$, where t is real, under the bilinear mapping

$$w = \frac{z - 2j}{z + j}$$

is a circle or a straight line. ***(5 marks)***

4 A vector field $\mathbf{A} = \mathbf{A}(x, y, z)$ is given by

$$\mathbf{A} = (4xy - z^3)\mathbf{i} + 2x^2\mathbf{j} - 3xz^2\mathbf{k}.$$

(i) Calculate div $\mathbf{A}$ and show that curl $\mathbf{A} = 0$. ***(6 marks)***

(ii) By evaluating both sides, verify that

$$\nabla^2\mathbf{A} = \text{grad div}\mathbf{A} - \text{curl curl}\mathbf{A}.$$

(11 marks)

(iii) Find a scalar field, $\Phi = \Phi(x, y, z)$, such that

$$\mathbf{A} = \text{grad}\Phi.$$

(8 marks)

End of Question Paper

Formula sheet

- The general formula for the residue at a pole z_0, of order m is

$$\frac{1}{(m-1)!}\lim_{z\to z_0}\left\{\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right\}$$

- Useful identities

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}; \quad \sin m\theta\cos n\theta = \frac{1}{2}[\sin(m+n)\theta + \sin(m-n)\theta]$$