The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2013-2014

Analytical Dynamics and Classical Field Theory — 3 Hours

Answer five questions. If you answer more than five questions, only your best five will be counted.

1 In this question, you are asked to derive D'Alembert's principle and subsequently derive the Euler–Lagrange equations from it.

(i) D'Alembert's principle is expressed mathematically in the form

$$\sum_{i=1}^{N} (\dot{\mathbf{p}}_i - \mathbf{F}_i^s) \cdot \delta \mathbf{r}_i = 0,$$

where $\mathbf{p}_i$ is the momentum of particle i, $\delta \mathbf{r}_i$ is the virtual displacement of particle i and $\mathbf{F}_i^s$ is the specified force acting on that particle. Derive this equation from Newton's law

$$\dot{\mathbf{p}}_i = \mathbf{F}_i^{\text{tot}},$$

where $\mathbf{F}_i^{\text{tot}}$ is the *total* force (i.e. the sum of specified and constraint forces) acting on particle i. What is the physical content of D'Alembert's principle? ***(6 marks)***

(ii) Derive the Euler–Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$$

from D'Alembert's principle, assuming that the masses of the particles are constant and the forces acting on them are conservative (and hence can be written in the form $\mathbf{F} = -\nabla V$, where V is a scalar function depending on the coordinates only). Here, $\mathcal{L} = T - V$ is the Lagrange function, where T is the total kinetic energy and V the total potential energy, q_i are the generalised coordinates and $\dot{q}_i = \mathrm{d}q_i/\mathrm{d}t$. ***(14 marks)***

2 (i) State Hamilton's principle, defining all quantities. ***(6 marks)***

(ii) Suppose that the Lagrangian is a function of the generalised co-ordinates q_i, the generalised velocities $\dot{q}_i = \mathrm{d}q_i/\mathrm{d}t$, the generalised accelerations $\ddot{q}_i = \mathrm{d}^2q_i/\mathrm{d}t^2$ and time t, i.e. $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i, \ddot{q}_i, t)$. Starting with Hamilton's principle (with both δq_i and $\delta \dot{q}_i$ vanishing at the start and end times), show that the Euler–Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial \mathcal{L}}{\partial \ddot{q}_i} = 0 \ .$$

(12 marks)

(iii) In general, of what order are the resulting differential equations if $\mathcal{L}$ depends on q_i, $\dot{q}_i$, $\ddot{q}_i$ and t? ***(2 marks)***

3 Consider a system of N particles, each with mass m_i (where the m_i are distinct), which move in a potential $V(x, y, z)$ such that the force acting on the particles is of the form $\mathbf{F} = -\nabla V$.

(i) Write down the Lagrange–function and determine the Hamilton–function H. Show that Hamilton–function H is the sum of kinetic energy and potential energy. ***(7 marks)***

(ii) Write down the equations of motion, using Hamilton's equations. ***(4 marks)***

(iii) Using the results obtained in (i) and (ii), show that the Hamilton function is constant, i.e. $\mathrm{d}H/\mathrm{d}t = 0$. ***(3 marks)***

(iv) Using Hamilton's equations, show that if $f = f(q_i, P_i, t)$ is an arbitrary function of co–ordinates, momenta and time, the following holds:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -\{H, f\} + \frac{\partial f}{\partial t} \ ,$$

where $\{...\}$ denote the Poisson brackets. ***(6 marks)***

4 You are given that the Lorentz–transformation between two inertial reference frames with co-ordinates x^μ (System A) and x'^μ (System B) is given by $\Lambda^\mu{}_\nu = \partial x'^\mu / \partial x^\nu$.

(i) What is the definition of the line element $\mathrm{d}s$ in Minkoswki space? What is the definition of proper time $\mathrm{d}\tau$? Show that the proper time is invariant under Lorentz transformations. ***(5 marks)***

(ii) Define the four–velocity u^μ and four acceleration a^μ. Show that $u^\mu u_\mu = c^2$ and $u^\mu a_\mu = 0$. ***(7 marks)***

(iii) How does the tensor $\rho^\mu{}_\nu$ transform under Lorentz–transformations? By explicit calculation, show that $\rho^\mu{}_\nu \sigma_\mu \epsilon^\nu$ is invariant under Lorentz–transformations, where σ^μ and ϵ^ν are arbitrary four–vectors. ***(8 marks)***

5 Consider the following Lagrangian $\mathcal{L}$, describing a vector field A^μ:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \eta_{\mu\nu} j^\mu A^\nu,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(i) Show that the Euler–Lagrange equations lead to

$$\partial_\mu F^{\mu\nu} = j^\nu \ .$$

Working in the Lorenz–gauge $\partial_\mu A^\mu = 0$, show that the Euler–Lagrange equations lead to

$$\eta^{\mu\nu} \partial_\mu \partial_\nu A^\alpha = j^\alpha,$$

where $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the metric tensor in Minkowski spacetime.

(15 marks)

(ii) Consider the transformation $A^\mu \to A^\mu + \partial^\mu \theta$, where θ is an arbitrary function of the coordinates x^ν. Show that the Lagrangian is invariant under this transformation if $\partial_\mu j^\mu = 0$. ***(5 marks)***

6 (i) State the contents of Noether's theorem. ***(2 marks)***

(ii) The following action is given, describing a scalar field:

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) = \int d^4x \left(\frac{1}{2} \eta^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - V(\phi) \right).$$

The potential $V(\phi)$ is given by

$$V(\phi) = \frac{1}{4} \lambda \phi^4.$$

Consider now the following transformations of the coordinates x^μ and field ϕ

$$\begin{aligned} x^\mu &\to e^\alpha x^\mu \\ \phi(x) &\to \phi(x) \exp(-\alpha), \end{aligned} \qquad (1)$$

where α is a constant. Show that the action is invariant under the transformations above.

(10 marks)

(iii) Find the associated Noether current. You are given that the Noether current can be written as

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \left[A^\nu(x) \partial_\nu \phi - F(\phi, \partial\phi) \right] - A^\mu(x) \mathcal{L},$$

for general infinitesimal transformations of the form (ϵ is a small parameter)

$$\begin{aligned} x'^\mu &= x^\mu + \epsilon A^\mu(x) \\ \phi'(x') &= \phi(x) + \epsilon F(\phi, \partial\phi) \end{aligned} \qquad (2)$$

Hint: Assume that α is small and expand the transformations (1) to bring them into the form (2). ***(8 marks)***

End of Question Paper