Data provided: Formula Sheet

MAS420

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2014–15

MAS420 Signal Processing — 2 hours

Attempt all FOUR questions.

1 (i) Plancherel's isometry theorem states that if f and g are signals with Fourier transforms F and G respectively, then

$$(f, g) = \frac{1}{2\pi}(F, G).$$

(a) Prove this result. ***(6 marks)***

(b) Deduce that

$$\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = 2\pi E(f),$$

where E denotes the energy of the signal. ***(1 mark)***

(ii) Consider the signal

$$f(t) = \text{sinc}^2(5t).$$

(a) Calculate and sketch the signal spectrum $F(\omega)$. ***(3 marks)***

(b) Calculate the energy of the signal. ***(3 marks)***

(c) For this signal verify that $\tau\Omega > \pi$, where Ω (rad/s) is the bandwidth and $\tau = \dfrac{E}{|f(0)|^2}$ is the equivalent rectangle resolution. ***(3 marks)***

(d) Find the Nyquist frequency (in Hz) of $f(t)$. ***(1 mark)***

(e) If $f(t)$ is sampled at three quarters of the Nyquist frequency and a signal $g(t)$ is constructed from these samples by sinc interpolation, with the aid of clear diagrams show that

$$g(t) = A \,\text{sinc}\left(\frac{15t}{2}\right) + B \,\text{sinc}^2\left(\frac{5t}{2}\right),$$

where A and B are constants, and find the values of A and B. ***(8 marks)***

2 (i) For the Hilbert space $L^2[0,3]$:

(a) Find $\|h\|$, where $h(t)$ is the signal $(3+it)^{-1}$. ***(3 marks)***

(b) Calculate the inner product $(t, e^{-i\omega t})$. ***(4 marks)***

(ii) The duality property of the Fourier transform states that if $g(t) \leftrightarrow G(\omega)$ then $G(t) \leftrightarrow 2\pi g(-\omega)$.

(a) Prove this result. ***(2 marks)***

(b) Use the duality property to find the Fourier transform of $\operatorname{sinc}(7t)$. ***(4 marks)***

(iii) (a) The function $f(t)$ is periodic with period 2π and is given by

$$f(t) = \begin{cases} 1 & : \quad 0 \le T \\ 0 & : \quad T < t < 2\pi. \end{cases}$$

Show that the complex Fourier series that represents $f(t)$ can be expressed as

$$f(t) = \frac{1}{2\pi}\left\{ T + \sum_{n=-\infty}^{-1} \frac{i}{n}\left[e^{-inT} - 1\right] e^{int} + \sum_{n=1}^{\infty} \frac{i}{n}\left[e^{-inT} - 1\right] e^{int} \right\}.$$

(6 marks)

(b) The signal $f(t)$ is passed through an ideal low-pass filter, $p_{\frac{\pi}{2}}(\omega)$, to yield a signal $\tilde{f}(t)$. Write $\tilde{f}(t)$ as a sine/cosine series and calculate the power of $\tilde{f}(t)$. ***(6 marks)***

3 (i) The convolution theorem states that if $F(\omega)$ and $G(\omega)$ are the Fourier transforms of the signals $f(t)$ and $g(t)$ respectively, then the Fourier transform of $f * g$ is FG, where $*$ denotes convolution.

(a) Prove the convolution theorem. ***(5 marks)***

(b) Use the convolution theorem to find the Fourier transform of the signal $h(t) = e^{-t}U(t) * e^{-5t}U(t)$, where $U(t)$ is the unit step function. ***(1 mark)***

(ii) (a) With the aid of clear diagrams, and without using Fourier transforms, find the convolution of the rectangular pulse, $p_1(t)$, with the function $g(t)$ which is defined by

$$g(t) = \begin{cases} t & : \quad 0 \le t \le 3 \\ 0 & : \quad t < 0 \quad \text{and} \quad t > 3. \end{cases}$$

(15 marks)

(b) Find the Fourier transform of $p_1(t) * g(t)$. ***(4 marks)***

4 (i) Define the following:

- a linear shift invariant (LSI) system;
- the system transfer function (STF), without any reference to the Fourier transform or the impulse response function;
- the impulse response function, without reference to the STF or convolution.

(4 marks)

(ii) (a) Prove that if S is the operator corresponding to a LSI system, and $\alpha \in \mathbb{C}$, then

$$S(e^{\alpha t}) = K(\alpha)e^{\alpha t},$$

where K is dependent on α, but independent of t. ***(5 marks)***

(b) Show how this gives rise to the concepts of system transfer function and frequency domain processing. ***(2 marks)***

(iii) (a) Show that

$$g(t) = \sum_{k=0}^{3} f(t-k)$$

is a linear shift invariant system. ***(3 marks)***

(b) Show that the STF of the $g(t)$ is given by

$$H(\omega) = e^{-\frac{3i\omega}{2}} \frac{\sin 2\omega}{\sin\left(\frac{\omega}{2}\right)}.$$

(6 marks)

(c) Find the response of the system to the input

$$g(t) = 3\cos 2t - 2\sin t,$$

giving your answer in terms of sine and cosine functions.

(5 marks)

End of Question Paper

Formula sheet

Function Definitions:

Rectangular pulse:

$$p_a(t) = \begin{cases} 1 & |t| \le a \\ 0 & |t| > a \end{cases}$$

Triangular pulse:

$$q_a(t) = \begin{cases} 1 - |t|/a & |t| \le a \\ 0 & |t| > a \end{cases}$$

Step function:

$$U(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Fourier Transform Pairs:

$$\begin{aligned}
p_a(t) &\longleftrightarrow 2a\,\mathrm{sinc}(a\omega) \\
q_a(t) &\longleftrightarrow a\,\mathrm{sinc}^2(a\omega/2) \\
\mathrm{sinc}(at) &\longleftrightarrow \frac{\pi}{a}p_a(\omega) \\
\mathrm{sinc}^2(at) &\longleftrightarrow \frac{\pi}{a}q_{2a}(\omega) \\
\mathrm{e}^{-at}U(t) &\longleftrightarrow \frac{1}{a+i\omega} \\
\delta(t) &\longleftrightarrow 1 \\
\delta(t-t_0) &\longleftrightarrow \mathrm{e}^{-i\omega t_0} \\
1 &\longleftrightarrow 2\pi\delta(\omega) \\
\mathrm{e}^{i\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega-\omega_0) \\
\mathrm{e}^{-t^2/2\sigma^2} &\longleftrightarrow \sigma\sqrt{2\pi}\mathrm{e}^{-\sigma^2\omega^2/2}
\end{aligned}$$

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

Duality theorem: If $f(t) \longleftrightarrow F(\omega)$ then $F(t) \longleftrightarrow 2\pi f(-\omega)$

Scaling: If $f(t) \longleftrightarrow F(\omega)$ then $f(at) \longleftrightarrow \frac{1}{|a|}F(\omega/a)$.

Translation: If $f(t) \longleftrightarrow F(\omega)$ then $f(t-t_0) \longleftrightarrow \mathrm{e}^{-i\omega t_0}F(\omega)$.

Frequency Shift: If $f(t) \longleftrightarrow F(\omega)$ then $\mathrm{e}^{i\omega_0 t}f(t) \longleftrightarrow F(\omega-\omega_0)$

Fourier Series: If $f_T(t)$ is periodic with period T then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\sigma t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-in\sigma t}\, dt$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\sigma t + b_n \sin n\sigma t)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t) \cos n\sigma t\, dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t) \sin n\sigma t\, dt$$

Parseval's Theorem: If V is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for V and $f = \sum_n c_n\phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Plancherel's Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t)g^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\, d\omega$$

Energy Theorem: If $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega$$

Convolution Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\, ds \longleftrightarrow F(\omega)G(\omega)$$

Product Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t)g(t) \longleftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega).$$