The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

Analytic Number Theory **2 hours 30 minutes**

*Answer **Question 1** and three other questions. You are advised **not** to answer more than three of the questions 2 to 5: if you do, only your best three will be counted.*

1 (i) Let χ be a character of the group $(\mathbb{Z}/N\mathbb{Z})^\times$. Define the Dirichlet L-series $L(s,\chi)$. Describe a region where $L(s,\chi)$ is analytic and convergent, dependent on χ. ***(3 marks)***

(ii) (a) List all characters of $(\mathbb{Z}/18\mathbb{Z})^\times$. ***(6 marks)***

(b) For the non-trivial real-valued character in your list, show explicitly the corresponding Dirichlet L-series does not vanish at $s = 1$. ***(2 marks)***

(c) Verify for all primes p:

$$\sum_{\chi} \chi(5)^{-1}\chi(p) = \begin{cases} 6 & \text{if } p \equiv 5 \bmod 18 \\ 0 & \text{otherwise} \end{cases}$$

where the sum is over all characters χ of $(\mathbb{Z}/18\mathbb{Z})^\times$. ***(6 marks)***

(iii) Using parts (i) and (ii) prove that there are infinitely many primes congruent to 5 mod 18.

(You may assume that for any character χ the sum $\sum_{p\nmid 18}\sum_{n\geq 2}\frac{\chi(p)^n}{np^{ns}}$ converges to a finite limit as $s \to 1$ and that $L(1,\chi) \neq 0$.) ***(8 marks)***

2 (i) Let $p_1, p_2, ..., p_n$ be the first n primes. For $x \geq 1$ let $N_n(x)$ be the number of integers $1 \leq k \leq x$ that are divisible **only** by the primes $p_1, p_2, ..., p_n$.

Show that

$$N_n(x) \leq 2^n \sqrt{x}.$$

Hence deduce that there are infinitely many primes. ***(6 marks)***

(ii) Consider the following infinite series over all primes:

$$S = \sum_p \frac{1}{p}.$$

(a) Show that S diverges:

(α) by using the result in part (i). ***(6 marks)***

(β) by using the Euler product expansion of $\zeta(s)$. (You may assume that $\zeta(\sigma) > 0$ for real $\sigma > 1$ and that $\zeta(\sigma) \to \infty$ as $\sigma \to 1^+$.) ***(7 marks)***

(b) Explain why the divergence of S proves that there are infinitely many primes. ***(2 marks)***

(iii) Does $\sum_p \frac{1}{p^2}$ converge? Justify your answer. ***(2 marks)***

(iv) Let $n \geq 2$ be a fixed integer. Are there infinitely many primes that are one less than an nth power? ***(2 marks)***

3 (i) Let f, g be be real functions. Define what it means for f and g to be asymptotic and state the Prime Number Theorem (PNT). (Your answer should include a definition of the prime counting function $\pi(x)$.)

(3 marks)

(ii) Fix a positive integer k. For $x \geq 1$ let $\pi_k(x)$ be the number of primes p such that $p^k \leq x$.

(a) Using the PNT show that

$$\pi_k(x) \sim \frac{k\sqrt[k]{x}}{\ln x}.$$

(4 marks)

(b) For $0 < a < b$ evaluate

$$\lim_{x\to\infty} \frac{\pi_k(bx)}{\pi_k(ax)}.$$

Hence, or otherwise, prove that for x large enough there exists a prime p such that $ax < p^k < bx$. ***(8 marks)***

(c) Let L be a positive integer. Using part (b) prove that there are are infinitely many primes p such that the decimal representation of p^k begins with L. ***(4 marks)***

(iii) Let $m \geq 1$ be an integer and let a be an integer coprime to m. For $x \geq 1$ let $\pi_{m,a}(x)$ be the number of primes $p \leq x$ that are congruent to a mod m.

Assuming that $\pi_{m,a} \sim \pi_{m,b}$ for any a and b coprime to m show that

$$\pi_{m,a}(x) \sim \frac{x}{\phi(m)\ln x}.$$

(You may use any valid properties of $\sim$ without proof.) ***(6 marks)***

4 (i) Let f and g be arithmetic functions.

(a) Define the Dirichlet series $D(s,f)$ and the Dirichlet convolution $f \star g$.

(b) Write down the relationship between the Dirichlet series for f, g and $f \star g$. ***(3 marks)***

(ii) (a) If f is completely multiplicative and $g_1, ..., g_n$ are arithmetic functions then show the following for $n \geq 2$:

$$f(g_1 \star g_2 \star ... \star g_n) = fg_1 \star fg_2 \star ... \star fg_n.$$

(b) Hence show that, for $a_1, a_2, ..., a_n \in \mathbb{Z}$:

$$N_{a_1} \star N_{a_2} \star ... \star N_{a_n} = N_{a_n}(N_{a_1 - a_n} \star N_{a_2 - a_n} \star ... \star N_{a_{n-1} - a_n} \star u)$$

(where $N_\alpha(n) = n^\alpha$ and $u(n) = 1$ for all n). ***(4 marks)***

(iii) Write down the Euler product expansion of $D(s,f)$ for multiplicative f. Hence show that if f is completely multiplicative then

$$D(s,f) = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

(4 marks)

(iv) Liouville's function is defined by

$$\lambda(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^{a_1 + a_2 + ... + a_k} & \text{if } n \text{ has prime factorisation } p_1^{a_1} p_2^{a_2} ... p_k^{a_k}. \end{cases}$$

(a) Where does $D(s, \lambda)$ converge absolutely? ***(1 mark)***

(b) Prove that λ is completely multiplicative and show that

$$D(s, \lambda) = \frac{\zeta(2s)}{\zeta(s)}.$$

(7 marks)

(c) Given an integer α let λ_α be the arithmetic function defined by $\lambda_\alpha(n) = \sum_{d|n} d^\alpha \lambda(d)$. Write $D(s, \lambda_\alpha)$ in terms of the Riemann zeta function. ***(6 marks)***

5 (i) Define the Riemann zeta function $\zeta(s)$ as a Dirichlet series and show convergence for $\mathrm{Re}(s) > 1$. State the Riemann Hypothesis.

(7 marks)

(ii) For $k \geq 0$ the Bernoulli polynomials $B_k(x)$ are given by the generating series

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

For $k \geq 1$ show the following:

(a) $\frac{d}{dx}(B_k(x)) = kB_{k-1}(x)$,

(b) $\int_0^1 B_k(x)dx = 0.$

(6 marks)

(iii) In the interval $[0, 1]$, $B_3(x)$ has a Fourier expansion of the form

$$B_3(x) = 12 \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{(2n\pi)^3}.$$

Using part (ii), or otherwise, find the Fourier expansion of $B_4(x)$ in the interval $[0, 1]$. Hence show that $\zeta(4) = \frac{\pi^4}{90}$. (It is given that $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$.)

(7 marks)

(iv) For real $a > 0$ and $\mathrm{Re}(s) > 1$ the Hurwitz zeta function is given by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}.$$

(a) Show that

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s).$$

(2 marks)

(b) If χ is a mod k Dirichlet character then show that

$$L(s, \chi) = k^{-s} \sum_{r=0}^{k} \chi(r)\zeta\left(s, \frac{r}{k}\right).$$

(3 marks)

End of Question Paper