The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014-15**

Fields **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$ (where $i^2 = -1$):

(a) $J_1 = \{a + bi\sqrt{7} : \ a, b \in \mathbb{Q}\}$, ***(4 marks)***

(b) $J_2 = \{a + bi + ci\sqrt{3} : \ a, b, c \in \mathbb{Q}\}$. ***(3 marks)***

(ii) Let $L = \mathbb{Q}(\sqrt{2}, i\sqrt{3})$.

(a) Show that $L = \mathbb{Q}(2\sqrt{2} + i3\sqrt{3})$. ***(5 marks)***

(b) Express the element $\dfrac{1}{1 + 2\sqrt{2} + i3\sqrt{3}}$ in the form

$$a + b\sqrt{2} + ci\sqrt{3} + di\sqrt{6}$$

for some $a, b, c, d \in \mathbb{Q}$. ***(4 marks)***

(iii) Let p be a prime number. Prove that

$$\phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is an irreducible polynomial in $\mathbb{Q}[x]$. ***(9 marks)***

2 (i) Let $K \subseteq L$ be a field extension. What is meant by saying that an element $b \in L$ is *algebraic* over K? ***(2 marks)***

(ii) Is the element $b = \sqrt[5]{\sqrt{2}+\sqrt{3}}$ algebraic over $\mathbb{Q}$? Justify your response and if the answer is 'yes' then find a polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(b) = 0$. ***(5 marks)***

(iii) Let $K \subseteq L$ be a field extension. Suppose that an element $c \in L$ is algebraic over K.

(a) Give a definition of the *minimal polynomial* $m(x) \in K[x]$ of the element c over K and prove that it is an irreducible polynomial over K. ***(6 marks)***

(b) Suppose that $n = \deg(m(x))$. Prove that the set of elements $1, c, c^2, \ldots, c^{n-1}$ form a basis for the vector space $K(c)$ over the field K. ***(9 marks)***

(c) Find the minimal polynomial $m(x) \in \mathbb{Q}(\sqrt{2})[x]$ of the element $c = \sqrt{2} + 2i\sqrt{3}$ over the field $\mathbb{Q}(\sqrt{2})$. ***(3 marks)***

3 (i) Define the content of a polynomial and find the content of the the polynomial $p = \sum_{n=10}^{100} n!x^n$. ***(4 marks)***

(ii) State Gauss's Lemma. ***(3 marks)***

(iii) Prove Gauss's Lemma. ***(9 marks)***

(iv) Show that the polynomial $f(x) = 3x^2+2015x+2$ is an irreducible polynomial over $\mathbb{Q}$. ***(9 marks)***

4 (i) Let L/K be a field extension of degree $n = [L : K]$ such that $L = K(a)$ for some element $a \in L$. Suppose that the field L contains all the roots, say $a_1 = a, a_2, \ldots, a_s$, of the minimal polynomial $m(x) \in K[x]$ of the element a over K. Find explicitly the group $G(L/K)$. ***(8 marks)***

(ii) Let $K = \mathbb{Q}$ and and $L = \mathbb{Q}(a)$ where $a = e^{\frac{2\pi i}{3}}$ and $i = \sqrt{-1}$. Using (i), or otherwise, find explicitly the group $G(L/K)$. ***(5 marks)***

(iii) Let F be a finite field. Define the characteristic of F. ***(2 marks)***

(iv) Let F be a finite field of characteristic $p > 0$. Show that $|F| = p^n$ for some $n \geq 1$. ***(4 marks)***

(v) Let $\overline{\mathbb{F}}_p$ be the algebraic closure of the field $\mathbb{F}_p$ that contains p elements, where p is a prime number. Let $F \subseteq \overline{\mathbb{F}}_p$ be a subfield that contains p^n elements. Show that F is unique. (One can use the Lagrange Theorem.) ***(6 marks)***

End of Question Paper