The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

Mathematics Core II **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

Section A

A1 Solve the following inequalities:

(i) $|2x - 3| < 7$; ***(2 marks)***

(ii) $\dfrac{1}{x} \leq 2x + 1$. ***(3 marks)***

A2 (i) Evaluate the limit $\lim_{x \to 2} \dfrac{x^3 - x^2 - x - 2}{x - 2}$ by using l'Hôpital's Rule. ***(2 marks)***

(ii) Evaluate the same limit by another method. ***(2 marks)***

A3 Let $f(x, y) = x^2y^2 + 5xy^4$.

(i) Let $c = f(1, 1)$. What is the value of c ? ***(1 mark)***

(ii) What is the equation of the curve in the surface $z = f(x, y)$ with constant y-value and passing through the point $(1, 1, c)$? ***(1 mark)***

(iii) What is the gradient of this curve at $(1, 1, c)$? ***(2 marks)***

(iv) Give the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 1, c)$. ***(3 marks)***

A4 Let z be a function of u and v where $u = 2x - y$ and $v = x - 2y$.

(i) Show that

$$\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = -3\frac{\partial z}{\partial v}.$$

(2 marks)

(ii) Assuming equality of mixed second-order partial derivatives, show that

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 3\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right).$$

(3 marks)

A5 Let R be the triangular region in the (x, y)-plane with vertices $(0, 0)$, $(1, 0)$ and $(1, 2)$. Calculate the integral

$$\int\int_R 7x^2y + 2xy^2\, dx\, dy.$$

(6 marks)

A6 What is the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 3^{n+1}}{(2n)!} z^n$? *(3 marks)*

Section B

B1 For the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & -1 & 5 \\ -3 & -2 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ -7 \end{pmatrix},$$

form an augmented matrix $(A|\boldsymbol{b})$ and reduce this matrix to row echelon form. Hence determine the full solution of this system. *(4 marks)*

B2 Determine a, b, c and d in such a way that

$$\begin{pmatrix} \frac{3}{5} & 0 & a \\ b & c & \frac{3}{5} \\ 0 & 1 & d \end{pmatrix}$$

is an orthogonal matrix. *(4 marks)*

B3 Show that the determinant

$$\begin{vmatrix} b+c & a & a^3 \\ c+a & b & b^3 \\ a+b & c & c^3 \end{vmatrix}$$

has a factor $(a+b+c)$ and then factorise it completely. ***(4 marks)***

B4 (i) Find the equation of the parabola with a directrix $x = 0$ and a focus $(6, 0)$. ***(3 marks)***

(ii) Find the equation of the ellipse with vertices $(\pm 5, 0)$ and foci $(\pm 4, 0)$. ***(3 marks)***

B5 If a point $\mathrm{P}(x_1, y_1)$ lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

show that the equation for the tangent at P is given by

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.$$

(4 marks)

B6 Consider a point P whose distances to the two fixed points $A(0, 0)$ and $B(b, 0)$ satisfy $AP : PB = 1 : r$, where $b, r > 0$ and $r \neq 1$.

(i) By deriving its equation show that the locus of P, that is the set of points with the above property, is a circle. ***(4 marks)***

(ii) Show that

$$\frac{2}{AB} = \frac{1}{AQ} + \frac{1}{AR},$$

where Q and R are the intersections of the circle with the x-axis. ***(4 marks)***

End of Question Paper