The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2014–2015

Vectors and Mechanics

2 Hours

Attempt all the questions. The allocation of marks is shown in brackets. The total number of marks available is 60.

1 Points P and Q have position vectors $\boldsymbol{p} = 3\boldsymbol{i} + 7\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{q} = -3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ respectively.

Find:

(i) The position vector of the mid-point of PQ;

(ii) The vector $\overrightarrow{PQ}$;

(iii) The parametric vector equation of the line PQ. ***(3 marks)***

2 (i) Given the vectors $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, find $\boldsymbol{a}.\boldsymbol{b}$.

Hence find the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, giving your answer in radians correct to three significant figures.

(ii) Given the vectors $\boldsymbol{u} = \boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k}$ and $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$, find $\boldsymbol{u} \times \boldsymbol{v}$.

Hence find a unit vector perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$. ***(5 marks)***

3 A line L has parametric vector equation

$$\boldsymbol{r} = (1 + 2\lambda)\,\boldsymbol{i} + (2 - \lambda)\,\boldsymbol{j} + (\lambda - 5)\,\boldsymbol{k}.$$

A plane Π has vector equation

$$\boldsymbol{r}.\,(-2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = 2.$$

(i) Find the point of intersection (if any) of the line L and the plane Π.

(ii) Explain why the line L is perpendicular to the plane Π. ***(5 marks)***

4 A particle is projected from the origin O with speed V at an angle θ above the horizontal. Air resistance can be ignored.

If the horizontal and vertical displacements of the particle at time t are x and z respectively, write down expressions for x and z in terms of t, V, θ and the acceleration due to gravity g.

A stone is thrown from a height of 1.5 m above flat ground, at an angle $\pi/4$ above the horizontal. It lands on the ground at a horizontal distance of 30 m from the point of projection. Ignoring air resistance and taking the acceleration due to gravity to be $g = 9.8$ m s^{-2}, find the speed of the stone when launched, giving your answer correct to two significant figures. ***(5 marks)***

5 At time t, the position vector of a moving particle of mass M is $\boldsymbol{r}(t)$, where

$$\boldsymbol{r}(t) = 3at\boldsymbol{i} + bt^4\boldsymbol{j}$$

and a and b are positive constants.

Find:

(i) The velocity and acceleration of the particle at time t;

(ii) The total force $\boldsymbol{F}$ acting on the particle at time t;

(iii) The kinetic energy of the particle at time t;

(iv) The work done by the force $\boldsymbol{F}$ on the particle during the period from $t = 0$ to $t = T$;

(v) The impulse of the force $\boldsymbol{F}$ on the particle during the period from $t = 0$ to $t = T$. ***(7 marks)***

6 At time $t = 0$ a particle A is at the origin and a particle B is at the point with position vector $5\boldsymbol{i} - 10\boldsymbol{j} - 12\boldsymbol{k}$ m.

Particle A moves with constant velocity $2\boldsymbol{i}$ m s^{-1} and particle B moves with constant velocity $4\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$ m s^{-1}.

Show that the least distance between A and B in the subsequent motion is $\sqrt{89}$ m. ***(4 marks)***

7 A car of mass M is travelling without slipping round a rough bend of radius R which is banked at an angle α to the horizontal.

Draw clear diagrams showing the forces on the car perpendicular to the driving direction if the car is travelling (a) at the minimum possible speed; (b) at the maximum possible speed. ***(3 marks)***

8 A particle of mass m moves along the x-axis with initial speed u.

The only force acting on the particle is a resistance force of magnitude mkv^2, where v is the speed of the particle and k is a constant.

Show that the equation of motion of the particle takes the form

$$\frac{1}{v}\frac{dv}{dx} = -k.$$

Find the speed of the particle when it has travelled a distance $x = \dfrac{\ln 2}{2k}$.

At this instant the particle collides and coalesces with a stationary particle of equal mass m. Find the speed of the new particle immediately after the collision. ***(7 marks)***

9 Two particles C and D, of masses m and $m/2$ respectively, are attached to the lower end of a light elastic string of stiffness $9mg/L$ where g is the acceleration due to gravity and L is the natural length of the string. The upper end O of the string is fixed.

Find the extension y of the string when the particles hang in equilibrium.

At time $t = 0$ the particle D falls off the end of the string. In the subsequent motion the distance OC is $x + \dfrac{10L}{9}$. Show that

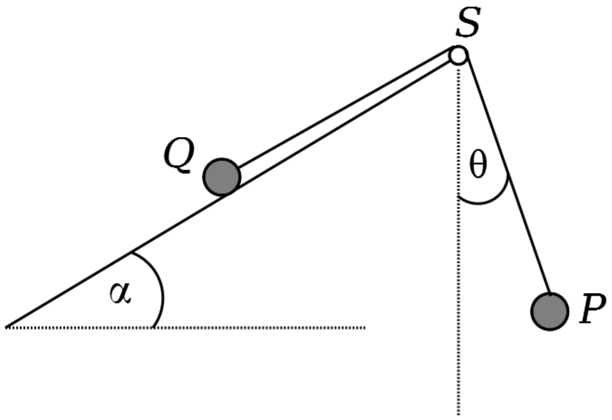
$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where ω is a positive constant which you should find in terms of g and L.

Find $x(t)$ in the subsequent motion. ***(8 marks)***

10 A particle P of mass M_1 is attached to one end of a light inextendible string. The other end of the string is attached to a second particle Q, of mass M_2. The particle Q is resting on a rough plane inclined at an angle α to the horizontal. The coefficient of friction between the particle Q and the inclined plane is $\tan\lambda$. Air resistance can be ignored.

The string passes over a smooth pulley S at the end of the plane. The pulley S exerts no forces on the system. The angle between the string SP and the downwards vertical is θ. The whole system is shown in the diagram below:



The particle Q is at rest. The particle P moves on an arc of a vertical circle with S at the centre so that the distance $SP = L$ where L is a constant. During the motion of P the angle θ varies between $-\beta$ and β.

(i) Draw a clear diagram showing the forces on the particles P and Q.

(ii) Show that the particle Q remains at rest if T, the magnitude of the tension in the string, satisfies

$$T \le M_2 g \frac{\sin(\alpha+\lambda)}{\cos\lambda}.$$

(iii) By considering the motion of the particle P, show that the tension in the string has magnitude

$$T = M_1 g\left[3\cos\theta - 2\cos\beta\right].$$

Hence find the maximum value of T during the motion of P.

(iv) Deduce that Q can remain at rest while P is moving if

$$M_1 \le M_2 \frac{\sin(\alpha+\lambda)}{\cos\lambda\left[3 - 2\cos\beta\right]}.$$

(13 marks)

End of Question Paper