Data provided: Formula sheet

MAS165

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014-2015**

Mathematics for Physicists **2 hours**

You should attempt ALL questions of this exam.

Section A

A1 A plane is given by the equation

$$3x + 2y + 4z = 21$$

and a line by the equation $\mathbf{r} = (1, 2, 3) + \lambda(1, 2, \mu)$, where λ is a real parameter and μ is a constant.

(i) Find μ so that the line does not intersect the plane. ***(4 marks)***

(ii) Using the value of μ you found in part (i), calculate the distance of the line to the plane. ***(4 marks)***

(iii) Find the direction of the line of intersection of the two planes $3x+9y-3z = 15$ and $3(x - y) + 2z = \pi$. ***(3 marks)***

A2 By explicit calculation, show that for a well-behaved vector field $\mathbf{F}$ the following identity holds: $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. ***(5 marks)***

A3 Stokes' theorem may be written:

$$\oint_C \mathbf{G}.d\mathbf{r} = \int_S (\nabla \times \mathbf{G}).\hat{\mathbf{n}}\, dS$$

Indicate whether the following statements about Stokes' theorem, as expressed here, are true or false

(i) The term $(\nabla \times \mathbf{G})$ is the curl of the vector field $\mathbf{G}$.

(ii) The surface S is surrounded by a closed line C.

(iii) $\hat{\mathbf{n}}$ is a unit vector parallel with the boundary C.

(iv) $\int_S dS$ is a surface integral, over the surface S.

(4 marks)

Section B

B1 (i) Consider the function

$$f(x, y) = \tan^{-1} \frac{y}{x}$$

Show that this equation obeys

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Hint: $\tan^{-1}$ (also called arctan) is the inverse function of the tan-function and you are given that

$$\frac{\mathrm{d} \tan^{-1} u}{\mathrm{d}u} = \frac{1}{1+u^2}.$$

(8 marks)

(ii) A scalar function is given as

$$\phi(x, y, z) = x^3 - zy\cos(x - z).$$

(a) Calculate the gradient of $\phi(x, y, z)$, i.e. calculate $\mathbf{V} = \nabla\phi$.
(3 marks)

(b) Using your result, calculate the divergence of $\mathbf{V}$. ***(3 marks)***

(c) Find the directional derivative of ϕ at the point $(0, 0, -\pi)$ in the y direction .
(6 marks)

B2 (i) A vector field is given by

$$\mathbf{V} = r^n \hat{\mathbf{r}} + \left(a + \frac{b}{r}\right) \hat{\boldsymbol{\theta}} + cz\hat{\mathbf{z}}$$

in cylindrical polar coordinates, where a, b and c are positive constants. Calculate the divergence and curl of the vector field, given that the divergence and curl may be expressed in cylindrical coordinates as

$$\nabla \cdot \mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta) + \frac{\partial}{\partial z}(V_z)$$

and

$$\nabla \times \mathbf{V} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_r & rV_\theta & V_z \end{vmatrix}$$

respectively. ***(10 marks)***

(ii) A hollow cylinder occupies the region $a \leq r \leq b$, $0 \leq \theta \leq 2\pi$, $0 \leq z < a$, where (r, θ, z) are cylindrical polar coordinates and a and b are positive constants. The mass density of the cylinder is given by

$$\rho = \frac{\rho_0}{ar} z^2, \tag{1}$$

where ρ_0 is a positive constant. Write down the volume element dV in cylindrical polar coordinates and find the mass of the cylinder.

(10 marks)

B3 (i) A sphere of radius a is charged with charge density $q_0 r^3/a^3$, where r is the distance from the centre of the sphere, located at the origin of co–ordinates, and q_0 is a constant. Find the total charge of the sphere. ***(10 marks)***

(ii) A magnetic field is given, in cylindrical polar coordinates (r, θ, z), as $\mathbf{H} = H_0 r^2 \hat{\boldsymbol{\theta}}/a^2$, with $r \leq a$, where H_0 and a are positive constants. The magnetic field vanishes for $r > a$. Evaluate

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{r},$$

where $\mathcal{C}$ is the circle $z = 0$, $r = R$, described in the anticlockwise sense for $R < a$.

(10 marks)

End of Question Paper