The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **2014–2015**

Analysis I **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper the words 'least upper bound' (lub) and 'supremum' may be used interchangeably and the words 'greatest lower bound' (glb) and 'infimum' may be used interchangeably.

1 Let C_0 denote the unit interval $[0,1]$. Remove from C_0 the 'middle 8/10s' and define

$$C_1 = [0,1] \setminus (0.1, 0.9) = [0.0, 0.1] \cup [0.9, 1.0].$$

For each C_n with $n \geq 1$ remove the middle 8/10s from each of the closed intervals in C_n, and let C_{n+1} be the union of the remaining intervals.

(i) Show that C_n consists of 2^n subintervals. **(3 marks)**

(ii) Find the sum of the lengths of the 2^n subintervals in C_n. **(4 marks)**

(iii) Write $C = \cap_{n=0}^{\infty} C_n$. Show that no open interval (a,b), where $0 \leq a < b \leq 1$, is a subset of C. **(3 marks)**

2 (i) State the *completeness property* for the real numbers $\mathbb{R}$. **(3 marks)**

(ii) Define what it means for a sequence (t_n) to be *monotone increasing.*

Prove that if (t_n) is a monotone increasing sequence which is bounded above, then it has a limit. **(5 marks)**

(iii) Show that the sequence (t_n) where $t_n = (1 + \frac{1}{n})^n$ is convergent. (You are not asked to find the limit.) **(7 marks)**

3 (i) State what is meant by the statement that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $a \in \mathbb{R}$. **(2 marks)**

(ii) State what is meant by the statement that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *differentiable* at a point $a \in \mathbb{R}$. **(2 marks)**

(iii) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$, then there is $\delta > 0$ and $K > 0$ such that

$$\text{if } |t-a| < \delta \text{ then } |f(t) - f(a)| \leq K|t-a|.$$

(7 marks)

(iv) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$, then f is continuous at a. **(3 marks)**

(v) Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} t^2 & t \geq 0; \\ -t^2 & t \leq 0. \end{cases}$$

Prove that f is differentiable and find $f'(t)$ for all $t \in \mathbb{R}$. Show that the derivative f' is not differentiable on $\mathbb{R}$. You do not need to differentiate from first principles when $t \neq 0$. **(5 marks)**

(vi) Find a function $g\colon \mathbb{R} \to \mathbb{R}$ such that the first and second derivatives of g exist on $\mathbb{R}$, but such that the second derivative is not differentiable on $\mathbb{R}$.

Justify your answer. **(6 marks)**

4 (i) Let (f_n) be a sequence of functions $f_n\colon [a,b] \to \mathbb{R}$.

(a) State what is meant by saying that (f_n) converges *pointwise* to a function $f\colon [a,b] \to \mathbb{R}$.

(b) State what is meant by saying that (f_n) converges *uniformly* to a function $f\colon [a,b] \to \mathbb{R}$.

(5 marks)

(ii) Let (f_n) be a sequence of continuous functions $f_n\colon [a,b] \to \mathbb{R}$ that converges uniformly to a function $f\colon [a,b] \to \mathbb{R}$. Prove that f is continuous.

Is the corresponding statement true for pointwise convergence? Prove it or give a counterexample. **(7 marks)**

(iii) Let (f_n) be a sequence of continuous functions $f_n\colon [a,b] \to \mathbb{R}$. If (f_n) converges uniformly to a function f, show that

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

(5 marks)

(iv) Compute the limits

$$\lim_{n\to\infty} \int_0^1 \frac{e^{t^4}}{n}\, dt \quad \text{and} \quad \lim_{n\to\infty} \int_1^2 t^{2-((\sin nt)/n)}\, dt,$$

justifying any procedures you use. **(8 marks)**

5 (i) Define the *radius of convergence* of a power series. **(3 marks)**

(ii) Find the radius of convergence of the power series

$$A(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

and prove that within the radius of convergence

$$A'(x) = \frac{1}{1-x^2}.$$

(7 marks)

6 (i) Define an *open set* in $\mathbb{R}^k$. **(3 marks)**

(ii) State a criterion for a function $f\colon \mathbb{R}^k \to \mathbb{R}$ to be continuous in terms of open sets. **(2 marks)**

(iii) Which of the following subsets of $\mathbb{R}^3$ are open? Justify your answers.

- $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 > 1\}$;
- $B = \{(x,y,z) \in \mathbb{R}^3 \mid x > 0, y > 0, z > 0\}$;
- $C = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = 0\}$;
- $D = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$.

(10 marks)

End of Question Paper