The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

MAS222 Differential Equations **2.5 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Write the general planar first order autonomous system for the following equation

$$\ddot{x} \; + \; \omega^2 x = 0,$$

where ω is a constant. ***(2 marks)***

Find the equilibrium point and its nature. ***(5 marks)***

Sketch the phase portrait for all three cases: (a) $\omega < 1$, (b) $\omega > 1$, and (c) $\omega = 1$. ***(5 marks)***

(ii) Find the equilibrium points of the following system

$$\dot{x} = x(3 - x - 2y), \quad \dot{y} = y(x - 1).$$

Using the eigenvalues, classify each of the equilibrium points. ***(9 marks)***

Sketch the phase portrait. ***(4 marks)***

2 (i) Find the general solution (up to first 5 terms) of

$$y'' + xy' + y = 0$$

around $x_0 = 1$, using the power series method. ***(12 marks)***

(ii) Write down the self-adjoint form of the differential equation

$$y'' - 4y' + \mu y = 0, \qquad 0 \leq x \leq 1.$$

The above differential equation is an eigenvalue problem for the constant μ with $y(0) = 0$ and $y'(1) = 0$.

State the orthogonality relation (in the form of a definite integral) satisfied by y_m and y_n, the eigenfunctions associated with the eigenvalues μ_m and μ_n respectively. ***(5 marks)***

(iii) It is given that $x = 0$ is a singular point of the differential equation:

$$x^2y'' - (2 + x^2)y = 0.$$

Show that $x = 0$ is a regular singular point of the differential equation. ***(2 marks)***

Use the Frobenius series expansion to show that the roots of the indicial equation are 2 and -1. ***(6 marks)***

3 (i) The function $u(x,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}\ ,$$

in the domain $0 < x < \pi$, $t > 0$, where k is a positive constant.

(a) Show that separable solutions of the form $X(x)T(t)$ satisfy the relation

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \alpha\,,$$

where α is a constant. Explain why α is a constant. ***(2 marks)***

(b) Assuming that $\alpha = -s^2$ (s is a positive real number), we have

$$X(x) = A\cos sx + B\sin sx,$$

where A and B are arbitrary constants. Determine the general solution for $u(x,t)$ with boundary conditions

$$u(0,t) = u(\pi,t) = 0 \text{ for } t \geq 0.$$

You may assume there are only trivial separable solutions when $\alpha \geq 0$. ***(8 marks)***

(c) Then find the solution for initial condition $u(x,0) = 4\sin(3x)$. ***(3 marks)***

(ii) Find the solution for the following inhomogeneous heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-kt}\sin(2x), \quad 0 < x < \pi,\ t > 0$$

with the same boundary and initial conditions as in (i):

$$u(0,t) = u(\pi,t) = 0 \text{ for } t \geq 0, \quad \text{and} \quad u(x,0) = 4\sin(3x) \text{ for } 0 < x < \pi.$$

k is a positive constant. ***(12 marks)***

4 (i) Find a set of characteristic coordinates $\xi(x,t)$ and $\eta(x,t)$ for the following equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} - 6\frac{\partial^2 u}{\partial t^2} + x\frac{\partial u}{\partial t} = x^2.$$

Note: find the characteristic coordinates only. You do NOT need to simplify the PDE.

(6 marks)

(ii) Find the solution for the following first order partial differential equation

$$\frac{\partial u}{\partial t} + 2x\frac{\partial u}{\partial x} = -u, \quad u = u(x,t),$$

with $u(x,0) = \sin x$. ***(9 marks)***

(iii) Consider the heat equation for $u(x,t)$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

in the domain $-1 \le x \le 1$, with boundary conditions

$$u(-1,t) = 0, \quad -3u(1,t) + \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=1} = 0.$$

For a separable solution $u(x,t) = X(x)T(t)$, you are **given** that

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \alpha,$$

where α is a constant. Assuming $\alpha = -s^2$ ($s > 0$ is a real number), show that s must satisfy the following relation when the solution is non-trivial:

$$s\cos(2s) = 3\sin(2s).$$

(10 marks)

End of Question Paper