The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2014–2015

Mathematics (Computational and Numerical Methods) — 2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) The exact value of $\sqrt[4]{5}$ is in the interval $(1, 2)$. Using the two starting values, $x_0 = 1$ and $x_1 = 2$, do 4 iterations of the secant method to find an approximation of $\sqrt[4]{5}$, stating your answer to an accuracy of 4 decimal places. (7 marks)

(ii) Show that finding the value of $\sqrt[4]{5}$ with the Newton-Raphson method leads to a recurrence relation of the form

$$x_{n+1} = \frac{1}{4}\left(3x_n + \frac{5}{x_n^3}\right).$$

(3 marks)

(iii) A square matrix A can be factorised such that $A = LU$, where L is a lower triangular matrix with unit diagonal elements and U is an upper triangular matrix. If all the inverse matrices exist for A, L and U, show that

$$A^{-1} = U^{-1}L^{-1}.$$

(2 marks)

(iv) Factorise the matrix

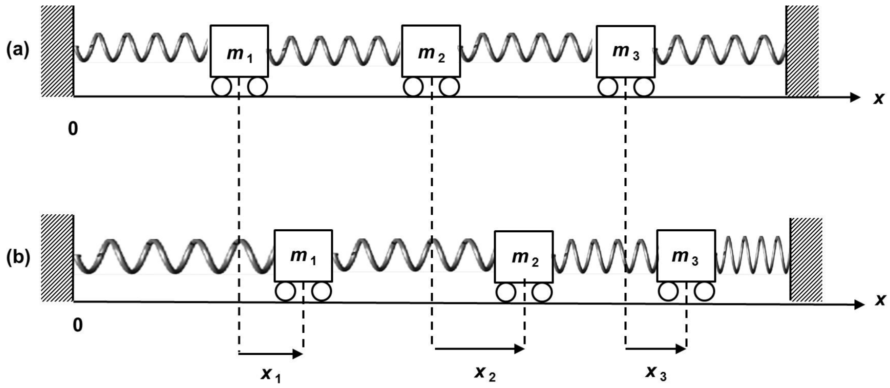
$$A = \begin{pmatrix} 2 & 8 & 5 \\ 1 & 6 & 8 \\ 1 & 3 & 2 \end{pmatrix}$$

into the product $A = LU$. Hence, using this LU factorisation, solve the following matrix equation for x, where

$$A\mathrm{x} = \mathbf{b}$$

and $\mathbf{b} = [1, 1, 1]^T$. (13 marks)

2 (i) Figure (a) shows three equal masses, m_1, m_2 and m_3, connected by four springs at their equilibrium positions. Figure (b) shows each mass displaced by an amount x_1, x_2 and x_3 respectively.



Assume the motions of the masses are undamped and governed by Hooke's law, i.e., the restoring force of each spring is $F = -k\Delta x$, where k is the spring constant and Δx is the change in spring length from the equilibrium value.

Take the value of k to be equal for all the springs and assume the oscillatory time dependence is $x_1 = X_1 \sin(\omega t)$, $x_2 = X_2 \sin(\omega t)$ and $x_3 = X_3 \sin(\omega t)$, where ω is the angular frequency and X_1, X_2 and X_3 are constants.

Form the eigenvalue problem of this system and hence, to an accuracy of 2 decimal places, find all the possible values of ω when $m_1 = m_2 = m_3 = 1$ kg and $k = 50$ N m^{-1}. (12 marks)

(ii) Using the composite Simpson's method evaluate

$$\int_1^5 \frac{1}{3} x \ln\left(\sqrt{x}\right)\, dx$$

to an accuracy of $\epsilon = 10^{-3}$. Give your final answer to an accuracy of 3 decimal places.

Hint: If a function $f(x)$ has four continuous derivatives on an interval (a, b) and this interval is divided into n subintervals, where n is an even positive integer, then the error bound for Simpson's method is given by

$$\left|E_n^S\right| \leq \frac{h^4}{180}(b-a)K,$$

where

$$h = \frac{b-a}{n}$$

and

$$K = \max_{a \leq x \leq b} \left| \frac{d^4 f(x)}{dx^4} \right|.$$

(13 marks)

3 (i) Fit a least squares quadratic, i.e., a polynomial of degree $n = 2$, to the data

x	1	2	3	4	5
f(x)	1.68	2.36	3.29	4.56	6.34

The system of equations arising should be solved using Gaussian elimination with partial pivoting where appropriate. Give the final polynomial coefficient values to an accuracy of 2 decimal places.

Hint: A polynomial of degree n can be expressed by the following sum,

$$P_n(x) = \sum_{j=0}^{n} a_j x^j.$$

In the least squares sense, a unique polynomial of degree n can be fitted to data points $(x_i, f(x_i))$, where $i = 0, 1, 2, \ldots, m$ and $m \geq n$. Assuming that the x_i values are free of errors, the normal equations used in the process of a least squares fit for a polynomial of degree n are

$$\sum_{i=0}^{m} \left(\sum_{j=0}^{n} a_j x_i^{j+k} \right) = \sum_{i=0}^{m} x_i^k f_i, \quad k = 0, 1, 2, \ldots, n.$$

(13 marks)

(ii) For function $y(x)$, derive the first five non-zero terms of the Taylor series solution to the ordinary differential equation,

$$y'' + x^2 y' + xy = 0,$$

subject to the initial conditions $y(1) = 0$ and $y'(1) = 1$.

Hint: The Taylor series for a function $y(x)$ around a point $x = x_0$ is given by

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \ldots$$

(7 marks)

(iii) Use Taylor series to derive the approximations,

$$y'(x_0) = \frac{y(x_0 + h) - y(x_0 - h)}{2h}$$

and

$$y''(x_0) = \frac{y(x_0 + h) - 2y(x_0) + y(x_0 - h)}{h^2},$$

showing that the leading terms of the error associated with these approximations are proportional to h^2. (5 marks)

4 (i) Sketch the feasibility region and find the maximum value of the function

$$f(x, y) = 2x + y,$$

subject to the constraints

$$-x + y \geq -1, \quad x + y \leq 6, \quad 4x + y \leq 12, \quad y \leq 5, \quad x, y \geq 0.$$

(10 marks)

(ii) A factory produces two types of laminate flooring which require three types of raw materials, P, Q and R. Each metre squared of type one flooring uses 1 kilogram of P, 2 kilograms of Q and 3 kilograms of R. Each metre squared of type two flooring uses 2 kilograms of P and 1 kilogram of R. The raw materials available each day are 380 kilograms of P, 320 kilograms of Q and 540 kilograms of R. The profit per metre squared for flooring type one is £1.50 and per metre squared for flooring type two is £1.20.

Formulate this into a linear programming problem and use graphical methods to determine the maximum possible daily profit. On the graph, clearly show the feasibility region and the line of constant revenue through the point of maximum daily profit. (15 marks)

End of Question Paper