The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

Probability Modelling **2 hours**

Candidates should attempt ***ALL*** *four questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–20; Q2–18; Q3–40; Q4–22)

1 A non-delayed renewal process has u_n, the probability of a renewal at time n, satisfying

$$u_n = \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(-\frac{1}{4}\right)^n$$

for $n \geq 0$.

(a) Find the generating function $U(s) = \sum_{n=0}^{\infty} u_n s^n$ for $0 \leq s < 1$. ***(5 marks)***

(b) Find the generating function $F(s) = \sum_{n=1}^{\infty} f_n s^n$, where f_n is the probability that the first renewal occurs at time n, for $0 \leq s < 1$, and hence deduce the values of f_1 and f_2. [You may use results from the course.] ***(8 marks)***

(c) Is this renewal process persistent or transient? Give a reason for your answer. [You may use results from the course.] ***(4 marks)***

(d) What is the probability that there is a renewal at time 2 which is not the first renewal after time 0? ***(3 marks)***

2 A discrete time Markov chain (X_n) has state space $S = \{1, 2, 3, 4, 5, 6\}$. The communicating classes are $\{1, 2, 3\}$, which is transient with period 3, $\{4\}$, which is persistent, and $\{5, 6\}$, which is persistent and aperiodic.

(a) Explain why state 4 must be aperiodic. ***(5 marks)***

(b) Give a possible transition matrix which fits this description. ***(13 marks)***

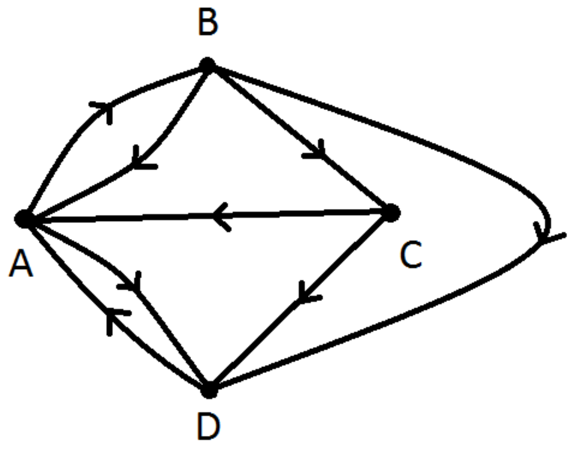

Figure 1: Network of web pages for question **3**

3 Consider the network of web pages $\{A, B, C, D\}$, shown in Figure 1, where an arrow between two pages represents a link between them in the indicated direction. A web crawler visits pages in turn, and, whenever it is at a given page, it chooses one of the links from that page to follow at random, with all choices being equally likely. Model the sequence of pages visited by the crawler as a Markov chain on $\{A, B, C, D\}$.

(a) Give the transition matrix of the chain, P. ***(4 marks)***

(b) For each page out of $\{A, B, C, D\}$, give an approximate probability that the crawler is at that page at some given time far in to the future, giving a careful explanation of your reasoning. [You may use results from the course.] ***(15 marks)***

(c) Find the probability that the crawler visits page C before page D, starting in

(i) A;

(ii) B. ***(8 marks)***

(d) Find the expected number of steps until the crawler visits page C, starting in

(i) A;

(ii) B;

(iii) D. ***(8 marks)***

(e) If the crawler is currently at page C, what is the expected number of steps until its next visit to page C? [You may use results from the course.] ***(5 marks)***

4 A company sets up a new web site. Letting time 0 be the time at which the site goes live, and measuring time in days, visits to the site occur as a Poisson process with variable rate $\lambda(t) = 4t$. Each visitor makes a purchase with probability $1/2$, independently of the time and the behaviour of other visitors.

(a) What is the distribution of the number of visitors to the site in the first five days after the site goes live? ***(4 marks)***

(b) Explain why the purchases made from the site occur as a variable rate Poisson process, and give the rate function. [You may use results from the course.] ***(5 marks)***

(c) What is the probability that there is exactly one purchase from the site made during the second day after the site goes live? ***(5 marks)***

(d) Let T_1 be the random variable defined as the time when the first purchase made from the site occurs. Find the distribution function of T_1. ***(8 marks)***

End of Question Paper