The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014-2015**

Mechanics and Fluids **2 hours**

Attempt all four questions. The allocation of marks is shown in brackets.

1 (i) A slim rod is aligned with the x-axis, extends from 0 to L, and has mass per unit length $\rho = \mathrm{e}^{-x/L}$. Determine the position of the centre of mass of the rod.
(7 marks)

(ii) A uniform laminar occupies the region bounded by $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq a^2$. *Using a change of coordinates*, find the position of the centre of mass.
(6 marks)

(iii) A laminar of uniform mass per unit area σ occupies the region bounded by $y \geq x^2$ and $y \leq 4$. By splitting the laminar into horizontal strips, determine the position of the centre of mass of the laminar and its moment of inertia about the x-axis.

Using your results and quoting a relevant theorem, calculate the moment of inertia about an axis parallel to the x-axis that passes through the centre of mass of the laminar.
(12 marks)

2 A point body of mass m experiences a force $\boldsymbol{F} = -\hat{\boldsymbol{r}}\, m\mu/r^2$, where r is the distance from the origin and μ is a constant. The path taken by the body is given by

$$\boldsymbol{r} = r\,\hat{\boldsymbol{r}}\,, \quad \text{where} \quad r = \frac{l}{1+\varepsilon\cos\theta}\,,$$

and l, ε are constants.

(i) Show that the force $\boldsymbol{F}$ can be recovered from the potential $V = -m\mu/r$.

(6 marks)

(ii) The unit vector $\hat{\boldsymbol{r}}$ may be expressed as $\hat{\boldsymbol{r}} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$, in terms of the Cartesian unit vectors. Find the analogous expression for $\hat{\boldsymbol{\theta}}$ and the relationship between $\partial\hat{\boldsymbol{r}}/\partial\theta$ and $\hat{\boldsymbol{\theta}}$. Using your result, calculate the work done by $\boldsymbol{F}$ along C, the section of the path from where $r(\theta)$ is minimum to where it is maximum, i.e.

$$W = \int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}\,.$$

(14 marks)

(iii) Show that your result in part (ii) is equal to the change in gravitational potential energy.

(5 marks)

3 (i) For the vector field $\boldsymbol{F} = 2xz\boldsymbol{i} + x^2\boldsymbol{k}$, calculate $\boldsymbol{\nabla}\cdot\boldsymbol{F}$, $\boldsymbol{\nabla}\times\boldsymbol{F}$, $\boldsymbol{\nabla}(\boldsymbol{F}\cdot\boldsymbol{F})$ and $(\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{F}$.

(6 marks)

(ii) *Using suffix notation,* and the identity

$$\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}\,,$$

show that for any $\boldsymbol{u}$ such that $\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = \boldsymbol{0}$, the following holds:

$$\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u}) = (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\,.$$

(7 marks)

(iii) State Gauss' Theorem and show that it holds for the field $\boldsymbol{F}$ of part (i) with the volume bounded by $0 \le z \le h$ and $x^2 + y^2 \le a^2$.

(12 marks)

4 A sphere of radius a is fixed and centred on the origin. It is surrounded by an incompressible fluid of density ρ.

The velocity of the fluid is given by a potential, $\boldsymbol{u} = \boldsymbol{\nabla}\psi$ where

$$\psi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta\,.$$

Given that in standard notation $h_1 = 1$, $h_2 = r$ and $h_3 = r\sin\theta$, find the velocity. Verify that the appropriate boundary condition is satisfied at $r = a$ and determine the flow far from the sphere.

(8 marks)

Show that the pressure on the surface of the sphere is given by

$$p = p_\infty + \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right),$$

where far from the sphere $p \to p_\infty$.

(5 marks)

Show that the z-component of the force exerted by the fluid on the sphere's surface S, is

$$-\int_S p\,\cos\theta\,\mathrm{d}S\,.$$

Perform the integral. Provide, in a few words, a physical interpretation of the result.

(12 marks)

End of Question Paper