The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014-2015**

Milestones in Applied Mathematics II **2 Hours**

Answer all four questions.

1 (i) An ultraviolet light source emits light of wavelength 2×10^{-7} m. An aluminium plate is situated close to the source. The photoelectric work function for aluminium is 6.74×10^{-19} J, and the mass of an electron is 9.11×10^{-31} kg. Using the value $h = 6.626 \times 10^{-34}$ J s for Planck's constant and the value 3×10^{8} m s^{-1} for the speed of light, calculate the energy of each photon of light emitted by the source and the maximum kinetic energy of electrons emitted by the aluminium when the ultraviolet light from the source is incident on the aluminium plate. ***(7 marks)***

(ii) A person of mass 70kg is walking at a speed of 6 km/hr. Calculate the de Broglie wavelength and frequency of this person.

(6 marks)

(iii) A quantum mechanical particle of mass m moves freely in the interval $[-a, 0]$ on the x-axis where $a > 0$. Assuming that the energy is positive, calculate energy levels and corresponding stationary state wave functions. (You do not need to normalise wave functions for this part.) ***(12 marks)***

2 (i) A beam with energy $\dfrac{\hbar^2 k^2}{2m} > 0$ is incident from large positive values of x on a potential well of the form

$$V(x) = \begin{cases} 0 & a < x < \infty; \\ -V & 0 < x < a; \\ \infty & x < 0; \end{cases}$$

where $V > 0$ is a positive constant. Show that a suitable solution of the time-independent Schrödinger equation has the form, when $x > a$:

$$\Psi(x) = A\left(e^{-ikx} + Be^{ikx}\right),$$

and calculate B. ***(15 marks)***

(ii) A quantum mechanical particle of mass m is described by the wave function $\psi_1(x,t) = Ae^{i(kx-\omega t)}$, where A is a complex constant, and k and ω are real constants. Calculate the probability density and current for this quantum particle and the direction of travel of this particle. ***(5 marks)***

(iii) By using the probability density obtained in part (ii), explain the uncertainty in the position and momentum of this particle. ***(5 marks)***

3 The state vector ψ is an eigenvector of the Hamiltonian operator H with energy eigenvalue E, so that

$$H\psi = E\psi.$$

The Hamiltonian H is given by

$$H = T + V,$$

where

$$T = \frac{P^2}{2m}, \qquad V = kX^2;$$

where k is a positive constant and m is the mass of the particle.

In this question you may use the result that, for any operator A,

$$\langle\psi|[H, A]\psi\rangle = 0.$$

(i) Calculate $[H, X]$ and $[H, P]$. ***(6 marks)***

(ii) By taking $A = X$, show that

$$E_\psi(P) = 0.$$

(3 marks)

(iii) Show that

$$[H, XP] = -i\hbar\frac{P^2}{m} + 2i\hbar kX^2,$$

and hence, using $A = XP$, show that

$$E_\psi(T) = E_\psi(V).$$

(6 marks)

(iv) Show that

$$E_\psi(T) + E_\psi(V) = E,$$

and hence that

$$E_\psi(T) = \frac{1}{2}E.$$

(5 marks)

(v) Deduce that

$$\Delta_\psi(P) = \sqrt{\frac{mE}{2}}.$$

(5 marks)

4 The Hamiltonian of a quantum system is given by

$$H = \hbar\omega \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}.$$

At time $t = 0$, the state of the system is given by

$$\psi(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(i) Find the energy eigenvalues of the system and the corresponding normalised eigenvectors. Hence, write down the normalised stationary states.

(10 marks)

(ii) Find the state of the system at time t. ***(5 marks)***

(iii) At time t, find the probability of observing the system to be in the state

$$\psi_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and

$$\psi_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(6 marks)

(iv) The operator A is given by

$$A = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}.$$

Calculate the expectation value of A at time t. ***(4 marks)***

End of Question Paper