Data provided: Formulae sheet

MAS340

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2014-2015

Mathematics (Computational Methods) — Two hours

Marks will be awarded for your best FOUR answers

1 (i) Classify the following differential equations as either elliptic, parabolic or hyperbolic.

(a) $\dfrac{\partial u}{\partial p} + 3\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial p^2} = u$ ***(1 mark)***

(b) $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial y \partial x} + \dfrac{\partial u}{\partial y} + 3 = -\dfrac{\partial u^2}{\partial y^2} - 2u$ ***(1 mark)***

(c) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y \partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial x} = 5$ ***(1 mark)***

(ii) Discuss the benefits/drawbacks of the three systems of numerical solutions, i.e. implicit, explicit and Crank-Nicolson. ***(6 marks)***

(iii) Consider the following differential equation

$$3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial x^2} + 42u = 0,$$

By using the explicit difference scheme, solve the differential equation between $0 \le x \le 1, 0 \le y \le 1$ for

$$\Delta x = h = \frac{1}{3}, \qquad \Delta y = k = \frac{1}{2},$$

and the following boundary conditions

$$u(1, y) = 0, \qquad u(x, 1) = 1 - x^2,$$

under the assumption that the solution is symmetric across both the x and y axes, *i.e.* $u(x, y) = u(-x, y) = u(x, -y)$. ***(15 marks)***

(iv) State the order of the error. ***(1 mark)***

2 (i) Discuss the advantages/disadvantages of LU factorisation over 'traditional' matrix inversion. ***(2 marks)***

(ii) Determine the L and U matrices for the following system

$$A\mathsf{x} = b, \qquad A = \begin{bmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$

(5 marks)

(iii) Determine L^{-1} and U^{-1} ***(4 marks)***

(iv) Hence find the column vector x ***(3 marks)***

(v) Find the LU decomposition for the following tri-diagonal matrix

$$M = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

(6 marks)

(vi) Using your results from part (v), or otherwise, show M^{-1} to be

$$M^{-1} = \frac{1}{63} \begin{bmatrix} 31 & -30 & 28 & -24 & 16 \\ -15 & 45 & -42 & 36 & -24 \\ 7 & -21 & 49 & -42 & 28 \\ -3 & 9 & -21 & 45 & -30 \\ 1 & -3 & 7 & -15 & 31 \end{bmatrix}$$

(5 marks)

3 Consider the following expression

$$f(x, y) = x^2 + \frac{4}{3}xy + y^2 + 6 \qquad (1)$$

(i) State the four expressions which allow for analytical classification of stationary points. ***(4 marks)***

(ii) Find and classify all stationary points of Equation (1). ***(3 marks)***

(iii) Apply the method of steepest descent, for one iteration, to Equation (1) starting from the point (1,1). ***(8 marks)***

(iv) Determine the Newton step for the function given in Equation (1) from a starting point of your choice. ***(7 marks)***

(v) Discuss the implications of your answer in part (iv). ***(3 marks)***

4 (i) Write down the three properties of cubic splines and why they are important. ***(3 marks)***

(ii) Write down the values for σ_0 and σ_n under the assumption of

$$f''(x_0) = 0, \qquad f'(x_n) = 0$$

(2 marks)

(iii) Using the conditions derived in (ii), determine the cubic spline between the following data points

x	0	$\pi/3$	$2\pi/3$	π
f(x)	1	1/2	$-1/2$	-1

(18 marks)

(iv) Determine the value of the cubic spline at f($5\pi/6$). ***(2 marks)***

5 (i) In a dynamic network problem, describe how you would represent the relationship between isolated nodes. ***(2 marks)***

(ii) In a dynamic network problem, describe how you would represent the relationship between stage jumping nodes. ***(2 marks)***

(iii) A ship is to be loaded with a selection of goods of 3 types. The total weight of the load must not exceed 20 tonnes. The goods have weights and values as shown in the following table:

Type	Weight	Price
1	2	10
2	5	24
3	7	37

Using the dynamic programming algorithm, construct an appropriate table to find the combination of goods which gives the highest value for the load on the ship. ***(21 marks)***

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$