The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014-2015**

Applicable Analysis **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

You may use the following results when answering questions on this paper.

Table of Laplace Transforms	
Function	*Laplace Transform*
$t^\alpha e^{bt} (\alpha > -1)$	$\dfrac{\Gamma(\alpha+1)}{(s-b)^{\alpha+1}}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$f(t)e^{bt}$	$F(s-b)$
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} f^{(k-1)}(0) s^{n-k}$
$tf(t)$	$-F'(s)$

1 (i) Define what is meant by the statement that the improper integral $\int_a^\infty f(x)\,dx$ exists. ***(2 marks)***

Prove, **from your definition,** each of the following statements:

(a) $\displaystyle\int_0^\infty \frac{2x}{(x^2+1)^2}\,dx$ exists;

(b) $\displaystyle\int_e^\infty \frac{(\ln x)^2}{x}\,dx$ does not exist.

(4 marks)

(ii) State, without proof, the Comparison Test for convergence and divergence of integrals of the form $\int_a^\infty f(x)\,dx$. Your statement should include conditions under which the results are valid. ***(4 marks)***

Determine whether

$$\int_0^\infty \frac{2x\sin x}{(x^2+1)^2}\,dx$$

converges or diverges, giving reasons for your answer. ***(3 marks)***

(iii) Prove that

$$\int_1^\infty \frac{\cos x}{\sqrt{x}}\,dx$$

converges.

Determine whether

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx$$

converges or diverges, giving reasons for your answers. ***(12 marks)***

2 (i) State, without proof, the theorem concerning differentiation of an integral of the form $\int_a^\infty f(x,y)\,dx$. Your statement should include conditions under which the result holds. ***(4 marks)***

Let

$$F(y) \quad = \quad e^{-y}\int_0^\infty e^{-x^2 y}\,dx \quad (y>0)\,.$$

Prove that F is differentiable on every interval $[c,d]$ with $0<c<d$. You may assume that $\int_0^\infty x^2 e^{-x^2}\,dx$ converges.

Show also that,

$$F'(y) \quad + \quad \left(1+\frac{1}{2y}\right) F(y) \quad = \quad 0 \qquad (*)$$

for $c \le y \le d$. ***(9 marks)***

Deduce that $(*)$ holds for all $y>0$. ***(1 mark)***

By solving the differential equation $(*)$, find an expression for $F(y)$ in terms of y valid for $y>0$. You may assume that $\int_0^\infty e^{-t^2}dt \quad = \quad \frac{1}{2}\sqrt{\pi}\,.$ ***(6 marks)***

(ii) Define the Γ function. ***(2 marks)***

Prove that

$$\int_0^\infty x^2\,e^{-x^2\sqrt{x}}\,dx \quad = \quad \tfrac{2}{25}\,\Gamma(\tfrac{1}{5})\,.$$ ***(3 marks)***

3 Define the Beta function. State, without proof, the relation between the Beta and Gamma functions. ***(3 marks)***

Prove that

$$B(x,y) \;=\; 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta \qquad (x>0,\ y>0)$$

and

$$B(x,y) \;=\; \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}}\, du \qquad (x>0,\ y>0).$$

(2 marks)

Prove each of the following, stating any standard results you need to use:

(a) $$\int_0^{\pi/2} \cos^2\theta \sqrt{\cot\theta}\, d\theta \;=\; \frac{3\sqrt{2}\,\pi}{8}\,;$$

(b) $$\int_0^{\infty} \frac{x\sqrt{x}}{(1+x^5)^2}\, dx \;=\; \frac{\pi}{10}\,.$$

(c) $$\int_{-\infty}^{\infty} \frac{e^{3x}}{(e^{2x}+1)^2} \;=\; \frac{\pi}{8}\,.$$ ***(16 marks)***

Show that

$$\int_0^{\pi} \sqrt{\sin\theta}\, d\theta \;=\; \frac{4}{\sqrt{2\pi}}\,[\Gamma(\tfrac{3}{4})]^2\,.$$ ***(4 marks)***

4 (i) Define what is meant by the statement that $\int_0^\infty f(t)e^{-st}\,dt$ has abscissa of convergence c. ***(2 marks)***

Find the abscissa of convergence of $\int_0^\infty \frac{t^2}{1+t^3}\; e^{-st}\,dt$, giving reasons for your answer. ***(5 marks)***

(ii) In each of the following cases, find the function continuous on $[0,\infty)$, with the given Laplace transform:

(a) $\dfrac{4}{4s^2-1} \quad (s > \frac{1}{2})\,;$

(b) $\dfrac{4s+2}{s^2+4} \quad (s > 0)\,;$

(c) $\dfrac{2s+2}{(s^2+2s+5)^2} \quad (s > 1)\,.$

(8 marks)

(iii) Suppose the functions f and g are continuous on $[0,\infty)$. Define the convolution $f * g$ and state, without proof, a relation between $L(f*g)$, $L(f)$ and $L(g)$. ***(3 marks)***

Using Laplace transforms, find the function f continuous on $[0,\infty)$ such that

$$\int_0^t u f(u)\, e^{t-u}\, du \quad = \quad t^2(1-e^{-t}) \quad (t > 0).$$ ***(7 marks)***

5 (i) Let $f : [0, \infty) \to \mathbb{R}$ be continuous and suppose that the Laplace transform $F = L(f)$ exists on (c, ∞) for some $c \in \mathbb{R}$. State, without proof, the formula giving $L\left(\dfrac{f(t)}{t}\right)$ in terms of F. Your statement should include sufficient conditions to ensure the validity of the formula. ***(2 marks)***

Show that

$$L\left(\frac{\sinh^2 t}{t}\right) = \frac{1}{4}\ln\left(\frac{s^2}{s^2-4}\right) \quad (s > 2)\,.$$ ***(11 marks)***

(ii) Using Laplace Transforms solve the differential equation

$$t\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + (2t+1)y = e^{-2t}$$

subject to the conditions $y(0) = 1, \quad y(1) = \dfrac{2}{e^2}$. ***(12 marks)***

End of Question Paper