The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

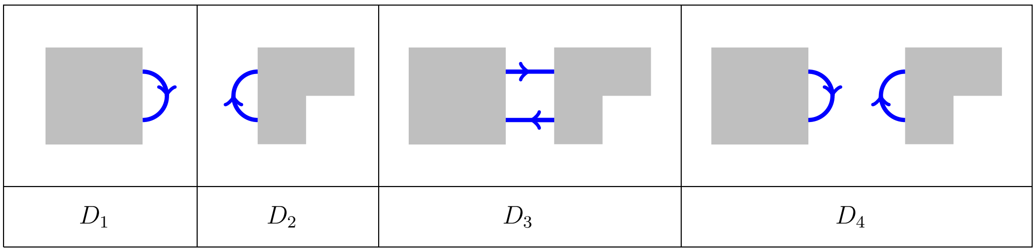
Knots and Surfaces **2 hours 30 minutes**

Answer all questions.

1 There is an invariant $g(D)$ for oriented link diagrams D, called the *Conway polynomial*. For the unknot we have $g(U_1) = 1$, and there is a skein relation $g(D_+) - g(D_-) = A\,g(D_0)$ (in the same context as the Jones polynomial skein relation).

Calculate $g(U_2)$, $g(H_-)$ and $g(T_-)$ (where U_2 consists of two unlinked circles, H_- is the negative Hopf link, and T_- is the negative trefoil). ***(13 marks)***

2 Consider four oriented link diagrams related as follows:



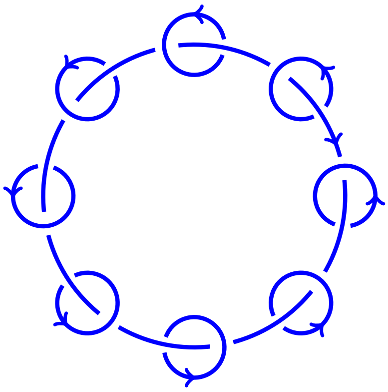
Use the skein relation to prove that $f(D_4) = -(A^2 + A^{-2})f(D_3)$ (where $f(D)$ denotes the Jones polynomial of D). Explain your logic carefully. ***(10 marks)***

3 Consider an oriented link diagram D with m crossings.

(a) Explain the definition of $\langle\langle D\rangle\rangle$, $\langle D\rangle$ and $f(D)$ (the unnormalised bracket, the Kauffman bracket and the Jones polynomial). ***(10 marks)***

(b) Let p denote the number of positive crossings, so there are $m - p$ negative crossings. Given a state S, let $\alpha(S)$ be the number of type A splitting markers, so there are $m - \alpha(S)$ markers of type B. By rewriting the relevant formulae in terms of p and $\alpha(S)$, show that $f(D)$ involves only even powers of A. ***(6 marks)***

4 Let N_n denote the necklace with n rings, as illustrated below.

$N_8 =$ 

Find the Jones polynomial of N_n. ***(16 marks)***

5 Let D be an oriented link diagram, and let D' be obtained from D by adding a positive loop as in Reidemeister move 1. Prove that $f(D') = f(D)$. ***(13 marks)***

6 (a) Let W be a surface word of the form $W = AxBC\overline{x}$. Explain geometrically why this represents the same surface as the word $W' = AxCB\overline{x}$. ***(5 marks)***

(b) Reduce the following surface words to standard form: ***(8 marks)***

$$W_1 = uvwuvw$$
$$W_2 = uvwwvu$$
$$W_3 = uvw\,\overline{u}\,\overline{v}\,\overline{w}$$
$$W_4 = uvwx\,\overline{u}\,\overline{v}\,\overline{w}\,\overline{x}$$

(c) Find the genus of the surface represented by the word $abc\overline{b}d\overline{c}e\overline{a}\,\overline{d}\,\overline{e}$. ***(6 marks)***

7 (a) Explain what is meant by a *covering pattern* for a surface, and explain how the Euler characteristic can be computed from a covering pattern. ***(5 marks)***

(b) If we fix an integer $n > 0$, then we can subdivide the unit square into smaller squares of side $1/n$, and this gives a covering pattern for the square. Verify that all these covering patterns give the same value for the Euler characteristic, independent of the choice of n. ***(8 marks)***

End of Question Paper