The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

MAS346 Groups and Symmetry **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Let G be a group. Define $\mathrm{Aut}(G)$ and prove that it is a subgroup of the group $S(G)$ of all bijections $f : G \to G$. [You may assume without proof that $S(G)$ is a group under composition of functions.] ***(6 marks)***

(ii) Let G be a group. Prove that for $a \in G$ the map $\omega_a : G \to G$ defined by

$$x \mapsto \omega_a(x) := axa^{-1}$$

is an element of $\mathrm{Aut}(G)$ and that the map $\omega_\bullet : G \to \mathrm{Aut}(G)$ given by $a \mapsto \omega_a$ is a homomorphism of groups. ***(5 marks)***

(iii) (a) Prove that $\mathrm{Aut}(\mathbf{Z}/n\mathbf{Z}) \cong (\mathbf{Z}/n\mathbf{Z})^*$.
[You may assume that $l_\bullet : \mathbf{Z}/n\mathbf{Z} \to \mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$ defined by $a \mapsto l_a$ is a bijection, where $l_a(x) = ax$ for all $x \in \mathbf{Z}/n\mathbf{Z}$.] ***(6 marks)***

(b) Express $\mathrm{Aut}(\mathbf{Z}/12\mathbf{Z})$ as a direct product of cyclic groups of prime power order. ***(2 marks)***

(iv) Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group where 1 is the identity of Q and the multiplication is given by the rules:

$$i^2 = j^2 = k^2 = -1,\ (-1)a = a(-1) = -a \text{ for all } a \in Q,$$

$$(-1)^2 = 1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

By considering the order of its elements describe 24 different automorphisms of Q. ***(6 marks)***

2 (i) Define the centre $Z(G)$ of a group G and prove that it is a normal subgroup. ***(4 marks)***

(ii) Give one example each of groups G_1, G_2, G_3 with

$$Z(G_1) = \{e\}, Z(G_2) = G_2 \text{ and } \{e\} \subsetneq Z(G_3) \subsetneq G_3.$$

(3 marks)

(iii) (a) Define the special orthogonal group SO_2 and the elements R_θ, S_θ of O_2. ***(3 marks)***

(b) By multiplying out matrices (and quoting relevant trigonometric identities) show that $S_\theta S_\phi = R_{\theta-\phi}$. ***(2 marks)***

(c) For $n > 2$ let $D_n = \{R^i_{2\pi/n} S^j_0 \text{ for } i = 1 \dots n \text{ and } j = 0, 1\}$. Determine $Z(D_n)$, distinguishing between n even and odd. You may use the identities $R_\theta R_\phi = R_{\theta+\phi}$, $R_\theta S_\phi = S_{\theta+\phi}$ and $S_\theta R_\phi = S_{\theta-\phi}$ without proving them. ***(5 marks)***

(iv) Let $\mathbf{F}_3 = \mathbf{Z}/3\mathbf{Z}$ be the field with 3 elements.

(a) Calculate the order of the group $\mathrm{SL}_3(\mathbf{F}_3)$. ***(3 marks)***

(b) Prove that the centre of $\mathrm{SL}_3(\mathbf{F}_3)$ is given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\}.$$

(5 marks)

3 (i) (a) Give the definition of the action of a group G on a set X. ***(3 marks)***

(b) Given a homomorphism $\phi : G \to S(X)$ explain how to define an action of G on X and prove that it satisfies the necessary axioms. ***(4 marks)***

(ii) Let $H < G$ be a subgroup. For $g \in G$ and $x \in G$ define $g * (xH) = (gx)H$.

(a) Show that this defines a group action of G on the set G/H of (left) cosets of H. ***(4 marks)***

(b) Define the homomorphism $\phi : G \to S(G/H)$ corresponding to the action in (a) and prove that

$$\ker(\phi) = \bigcap_{x \in G} xHx^{-1}.$$

(5 marks)

(iii) (a) Draw a 2-dimensional shape whose symmetry group is $D_3 \cong S_3$. ***(2 marks)***

(b) Prove that the direct symmetry group Dir(Dodec) of the regular dodecahedron has 60 elements by calculating the size of the orbit and stabilizer of a chosen face. ***(3 marks)***

(c) Arguing geometrically determine how many conjugacy classes there are in Dir(Dodec) of elements of order 5. [You may assume that all elements of order 5 are given by rotations about axes through centres of faces.] ***(4 marks)***

4 (i) State the Sylow theorems. You should carefully define all the terms and notation used. ***(5 marks)***

(ii) (a) Give the definition of a simple group. ***(2 marks)***

(b) Show that there is no simple group of order 224 by considering an appropriate group action. ***(4 marks)***

(c) By considering the order of elements show that a group of order p^2q with p, q distinct primes cannot be simple if there are p^2 Sylow q-subgroups. ***(5 marks)***

(iii) Determine the number of Sylow 3-subgroups of A_5. ***(4 marks)***

(iv) Let G be a group of order p^2q^2 for prime $p < q$. Prove that if G is not of order 36 then G has a normal q-Sylow subgroup. ***(5 marks)***

End of Question Paper