The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

MAS350 Measure and Probability **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) (a) Let S be a set. List the properties that Σ must have for (S, Σ) to be a *measurable space*. ***(3 marks)***

(b) Explain what is meant by a *measure* on (S, Σ). ***(2 marks)***

(ii) Let α and β be positive real numbers, and m_1 and m_2 be measures on (S, Σ).

(a) Show that $\alpha m_1 + \beta m_2$ is a measure on (S, Σ), where for each $A \in \Sigma$,

$$(\alpha m_1 + \beta m_2)(A) = \alpha m_1(A) + \beta m_2(A).$$

(4 marks)

(b) If $0 < m_1(A) < \infty$ for some $A \in \Sigma$, explain why αm_1 cannot be a measure when $\alpha < 0$. ***(1 mark)***

(c) Suppose that m_1 and m_2 are both probability measures. Give a condition on α and β so that $\alpha m_1 + \beta m_2$ is also a probability measure. ***(3 marks)***

(iii) In probability theory, the Poisson distribution of mean $\lambda > 0$ is defined for each $n \in \mathbb{Z}_+$ by

$$p_n = \frac{\lambda^n e^{-\lambda}}{n!}.$$

Explain briefly how these numbers can be used to construct a probability measure, stating clearly what (S, Σ) is in this case. Explain also how the numbers p_n can be obtained from the measure. ***(4 marks)***

(iv) (a) If $A, B, C \in \Sigma$ with $C \subset B \subset A$, deduce that

$$m(B \cap (A - C)) = m(B) - m(C).$$

(3 marks)

(b) Calculate the Lebesgue measure of the Borel subset E of $\mathbb{R}$ defined by

$$E = \left[0, \frac{1}{2}\right] \cap \left\{ [0,1] - \bigcup_{n=1}^{\infty} \left(\frac{1}{7^{n+1}}, \frac{1}{7^n} \right) \right\}.$$

(5 marks)

2 (i) (a) Explain what it means for a subset of $\mathbb{R}$ to be *open*. ***(1 mark)***

(b) Suppose that a mapping $f : \mathbb{R} \to \mathbb{R}$ is (I) *continuous*, (II) *measurable*. How would you describe these properties using open sets? ***(2 marks)***

(c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then it is measurable. ***(2 marks)***

(ii) Explain why the following functions from $\mathbb{R}$ to $\mathbb{R}$ are measurable, quoting any theorems that you use from the course:

(a) $f(x) = \cos(x)e^{-ax}$, where $a > 0$, ***(1 mark)***

(b) $f(x) = \cos(x)\mathbf{1}_{[0,1]}(x) + \sin(x)\mathbf{1}_{[1,2]}(x)$, ***(2 marks)***

(c) $f(x) = \tan^{-1}(\mathbf{1}_{(-5,1)}(x))$. ***(3 marks)***

(iii) Let (S, Σ) be a measurable space and (f_n) be a sequence of bounded measurable functions from S to $\mathbb{R}$.

(a) Define $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ and explain why these functions are measurable. ***(5 marks)***

(b) Suppose that $\liminf_{n\to\infty} f_n = \limsup_{n\to\infty} f_n$ *almost everywhere*. What can you say about the existence and measurability of $\lim_{n\to\infty} f_n$? ***(5 marks)***

(c) Define a sequence of functions (f_n) from $\mathbb{R}$ to $\mathbb{R}$ by

$$f_n(x) = \begin{cases} \left(1 + \dfrac{x}{n}\right) & \text{if } x \in (0,1) \\ 0 & \text{if } x \notin (0,1). \end{cases}$$

Is each f_n bounded and measurable? Does $\lim_{n\to\infty} f_n(x)$ exist for all $x \in \mathbb{R}$, and if so, does the limit give rise to a measurable function? ***(4 marks)***

3 Throughout this question (S, Σ, m) is a measure space.

(i) Explain what it means for a measurable function $f : S \to \mathbb{R}$ to be *integrable*. ***(1 mark)***

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = 3\mathbf{1}_{[-9,1)}(x) - 2\mathbf{1}_{[1,2)}(x) + 4\mathbf{1}_{[2,5)}(x) - 7\mathbf{1}_{[5,9]}(x).$$

(a) Explain why f and f^2 are measurable. ***(3 marks)***

(b) Show that f^2 is integrable by explicitly calculating $\int_{\mathbb{R}} f^2 dm$. ***(3 marks)***

(c) Obtain the inequality $\int_{\mathbb{R}} |f|^2 dm > \int_{\mathbb{R}} |f| dm$, and hence show that f is integrable. ***(3 marks)***

(d) Compute $\int_{\mathbb{R}} f dm$. ***(1 mark)***

(iii) (a) Let $f : S \to \mathbb{R}$ be a non-negative, measurable function. Explain why $\int_S f dm \geq 0$. ***(6 marks)***

(b) If f and g are non-negative, measurable functions from S to $\mathbb{R}$ with $f(x) \geq g(x)$ almost everywhere, deduce that $\int_S f dm \geq \int_S g dm$. ***(2 marks)***

(c) If $f : S \to [0, 1]$ is integrable, what can you say about the integrability of f^2? ***(2 marks)***

(iv) If $g : \mathbb{R} \to \mathbb{R}$ is integrable, show that f also is, where

$$f(x) = g(x)\mathbf{1}_{(0,\infty)}(x)(1 - e^{-\lambda x}),$$

and $\lambda > 0$. ***(4 marks)***

4 Throughout this question (S, Σ, m) is a measure space.

(i) State the *monotone convergence theorem*, and use it to prove *Fatou's lemma*: if (f_n) is a sequence of non-negative measurable functions then

$$\liminf_{n\to\infty} \int_S f_n dm \geq \int_S \liminf_{n\to\infty} f_n dm.$$

(8 marks)

(ii) Suppose that the sequence (f_n) of (i) converges pointwise to f. Deduce that

$$\limsup_{n\to\infty} \int_S f_n dm \geq \int_S f dm.$$

(3 marks)

(iii) Let $f : S \to \mathbb{R}$ be a non-negative measurable function and define a sequence (f_n) of functions by $f_n(x) = \min\{f(x), n\}$ for all $x \in S, n \in \mathbb{N}$. Deduce that

$$\lim_{n\to\infty} \int_S f_n dm = \int_S f dm.$$

(5 marks)

(iv) State the *dominated convergence theorem*, and use it to find

$$\lim_{n\to\infty} \int_0^\infty \frac{1 - e^{-nx}}{(1 + x^2)(1 - e^{-2nx})} dx.$$

(9 marks)

5 Throughout this question, $(\Omega, \mathcal{F}, P)$ is a probability space.

(i) Let (A_n) be a sequence of events in $\mathcal{F}$. Define the events $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$, and deduce that

$$P\left(\liminf_{n\to\infty} A_n^c\right) = 1 - P\left(\limsup_{n\to\infty} A_n\right)$$

(6 marks)

(ii) State both parts of the *Borel–Cantelli lemma*, and prove the part that does not require an independence assumption. ***(6 marks)***

(iii) Consider a sequence of (independent) rolls of a fair die. Deduce that the run 1412 appears infinitely often. ***(7 marks)***

(iv) Let (X_n) be a sequence of random variables which is such that for all $\epsilon > 0, P\left(\limsup_{n\to\infty}(|X_n| \geq \epsilon)\right) = 0$. What can you say about the asymptotic behaviour of X_n as $n \to \infty$? ***(6 marks)***

End of Question Paper