Data provided: Formula sheet

MAS424

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2014–2015

Differential Equations: Case Studies in Applied Mathematics — **2 hours**

Attempt all questions.

1 Consider a system of equations

$$\begin{aligned} \dot{x} &= y + x^3(4x^2 + y^2 - 4), \\ \dot{y} &= -4x + x^2 y(4x^2 + y^2 - 4). \end{aligned} \qquad (*)$$

(i) Show that the coordinate origin is a critical point of this system, and it is a centre. ***(5 marks)***

(ii) Use Theorem 1 from the Formula Sheet to show that system $(*)$ does not have periodic solutions in the region $x < -1$. ***(5 marks)***

(iii) Use the variable substitution

$$x = r\cos\theta, \qquad y = 2r\sin\theta,$$

to obtain the system of equations for r and θ,

$$\dot{r} = 4r^3(r^2 - 1)\cos^2\theta, \qquad \dot{\theta} = -2. \qquad (\dagger)$$

Use this result to show that system $(*)$ has a limit cycle solution defined by $4x^2 + y^2 = 4$. ***(6 marks)***

(iv) Linearize system $(\dagger)$ near the limit cycle solution $r = 1$. Solve this linearized system for arbitrary initial conditions. Use this solution to show that the limit cycle is unstable.
Hint: You may use the trigonometric identity $2\cos^2\theta = 1 + \cos 2\theta$.
(9 marks)

2 A steel plate has the shape of a square with the side d. Its thickness is $h \ll d$. The plate has been heated to temperature T and then put on a steel table. The table temperature is $T_0 < T$. You can assume that the temperature of the lower surface of the plate becomes equal to T_0 instantaneously and then remains at this temperature. You can also assume that the heat flux from the plate into the air is very small, so the upper boundary of the plate can be considered as thermally insulated. This implies that the temperature gradient at the upper plate is zero. Finally, you can assume that far from the side boundaries the temperature varies in the vertical direction only, so the temperature inside the plate, θ, satisfies the one-dimensional diffusion equation

$$\frac{\partial \theta}{\partial t} = a^2 \frac{\partial^2 \theta}{\partial z^2}, \qquad (*)$$

where the constant a^2 denotes the thermal diffusivity.

(i) Explain why, in Cartesian coordinates x, y, z with the z-axis vertical and the xy-plane coinciding with the table surface, the boundary conditions are

$$\theta = T_0 \quad \text{at} \quad z = 0; \qquad \frac{\partial \theta}{\partial z} = 0 \quad \text{at} \quad z = h.$$

(2 marks)

(ii) Find the constants A and B such that $\Theta = \theta + Az + B$ satisfies the homogeneous boundary conditions

$$\Theta = 0 \quad \text{at} \quad z = 0; \qquad \frac{\partial \Theta}{\partial z} = 0 \quad \text{at} \quad z = h.$$

Show that Θ satisfies equation $(*)$. What is the initial condition for Θ?

(4 marks)

(iii) Solve the initial value – boundary value problem for Θ. Then calculate the profile of the temperature θ in the plate far from the side boundaries at time t.

$\left(\text{You can use without proof the identity } \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{1+2n} \sin \frac{\pi(1+2n)z}{2h} = 1\right)$

(19 marks)

3 Consider the equation

$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial}{\partial x}\left((1+e^u)\frac{\partial u}{\partial x}\right),$$

which implies a density-dependent diffusion rate with logistic population growth. Physically, this equation describes a population which disperses more rapidly as the population grows.

(i) Using our standard technique of assuming a wave solution $u(x,t) = U(z)$ with $z = x - ct$, $c > 0$, write down the second-order ordinary differential equation for the wave profile $U(z)$. ***(3 marks)***

(ii) By introducing the variable $V = U'$ reduce the second-order ode to the system of two first-order odes. ***(3 marks)***

(iii) Write down the two critical points in the (U, V) plane, and classify them. Assume that $c \neq 2\sqrt{2}$. ***(16 marks)***

(iv) On the basis of this analysis conclude what are the limiting values of $U(z)$ corresponding to the travelling wave as $z \to \infty$ and $z \to -\infty$? Given that U represents the number of species, why do you think that the case $0 < c < 2\sqrt{2}$ is unphysical? ***(3 marks)***

4 (i) Prove that, if f is independent of x, i.e. $f = f(y, y')$, then

$$f - y'\frac{\partial f}{\partial y'} = \text{const}$$

is a first integral of the Euler-Lagrange equation. ***(4 marks)***

(ii) The speed of light in empty space is c. In general, in a transparent medium light propagates with speed $v < c$. The refractive index of the medium is $n = c/v$. The light propagates from point A to point B with the variable refractive index n. In Cartesian coordinates x, y the coordinates of these points are $A(0, 0)$ and $B(a, b)$, where $a > 0$ and $b > 0$.

(a) You are given that the light propagates along the ray $y = y(x)$, where $y(0) = 0$ and $y(a) = b$. Show that the propagation time is given by

$$T = \frac{1}{c}\int_0^a n(x, y)\sqrt{1 + y'^2}\, dx.$$

(4 marks)

(b) Fermat's principle states that the path taken between two points by a ray of light is the path that can be traversed in the least time. You are given that n is independent of x. Using Fermat's principle show that the ray $y = y(x)$ is defined by the equation

$$\sqrt{1 + y'^2} = Hn(y),$$

where H is an arbitrary constant. ***(4 marks)***

(c) You are given that $n(y) = \sqrt{1 + y/d}$, where d is a positive constant. You are also given that, at $x = 0$, the angle between the ray $y = y(x)$ and the x-axis is 45°. Show that $H = \sqrt{2}$. Then find the ray equation $y = y(x)$. Use this result to express b in terms of a and d. ***(9 marks)***

(d) Calculate the propagation time T. ***(4 marks)***

End of Question Paper

List of Basic Formulae and Theorems

Theorem 1: If a periodic solution of the system of equations

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

exists in a simply connected region, then $f_x + g_y = 0$ somewhere in that region.

Corollary: There are no periodic solutions in any simply connected region where $f_x + g_y \neq 0$ everywhere.

Theorem 2: The orbit $\mathcal{C}$ of a periodic solution must enclose at least one critical point.

Orthogonality conditions for trig functions

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{when } \; m \neq n.$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

Extremals of functional

$$J[y] = \int_{x_0}^{x_1} f(y, y', x) \, dx$$

are the solutions to the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$