The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

Galois Theory **2 hours 30 minutes**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

Throughout the paper K denotes a field. A ring is always assumed to be commutative and to contain an element $1 \neq 0$.

1 (i) Consider a reduced cubic equation

$$u^3 + pu + q = 0 \qquad (*)$$

where $p, q \in \mathbb{R}$.

Write $u = \sqrt[3]{y} + \sqrt[3]{z}$ where $y, z \in \mathbb{C}$.

(a) Obtain expressions for p and q in terms of y and z.

(b) Show that z is a solution of the quadratic equation $z^2 + qz - \frac{p^3}{27} = 0$.

(c) Obtain a formula for a solution u of the reduced cubic in terms of p and q. **(10 marks)**

(ii) Let K be a field and let L be an extension of finite degree. Let M be a subring of L with $K \subseteq M$. Prove that M is a subfield of L. **(8 marks)**

(iii) Let A be an abelian subgroup of Σ_n that acts transitively on the set $N = \{1, 2, \ldots, n\}$. Show that if $\sigma \in A$ and $\sigma(i) = i$ for some $i \in N$, then σ is the identity. Deduce the value of $|A|$. **(7 marks)**

2 (i) Let $\varphi\colon\ R \to S$ be a morphism of rings, and let I be an ideal in R such that $\varphi(a) = 0$ for all $a \in I$.

Show that there is a unique morphism $\overline{\varphi}\colon\ R/I \to S$ with $\overline{\varphi}\circ\pi = \varphi\colon\ R \to S$, where $\pi\colon\ R \to R/I$ is the canonical morphism.

Further, show that if φ is surjective and $\ker(\varphi) = I$ then $\overline{\varphi}$ is an isomorphism. **(10 marks)**

(ii) Let K be a field and let $\varphi\colon\ K \to L$ and $\psi\colon\ K \to M$ be homomorphisms of fields, with $[L : K]$ and $[M : K]$ finite.

(a) Define the set $E_K(\varphi, \psi) = E(\varphi, \psi)$. **(1 mark)**

(b) Now suppose that $L = K'(\alpha)$ where $K' = \varphi(K)$ and $\alpha \in L$. Write $q(t) = \min(\alpha, \varphi) \in K[t]$.

Show that there is a bijective correspondence between $E(\varphi, \psi)$ and the set R of roots of $\widetilde{\psi}(q)$ in M. Here $\widetilde{\psi}(q)$ is the polynomial in $M[t]$ obtained by mapping the coefficients of q into M by ψ.

You may use, if you wish, the result that the map

$$\overline{\chi}\colon\ K[t]/(K[t].q(t)) \to L$$

given by

$$\overline{\chi}\left(\sum_i a_i t^i + K[t].q(t)\right) = \sum_i \varphi(a_i)\alpha^i,$$

is a field isomorphism. **(14 marks)**

3 (a) Let $\psi\colon\ K \to L$ be a homomorphism of fields of finite degree. It is said to be *normal* if for every irreducible polynomial $f(t) \in K[t]$ such that $\widetilde{\psi}(f) \in L[t]$ has a root in L, $f(t)$ is properly split by ψ.

State one of the three equivalent characterizations of a normal homomorphism which were established in the course. **(3 marks)**

(b) State the Main Theorem of Galois Theory, concerning intermediate subfields and intermediate subgroups. Be careful to define the notation you use. **(7 marks)**

(c) Let L be a normal extension of $\mathbb{Q}$ such that $G(L/\mathbb{Q})$ is isomorphic to $C_2 \times C_2$.

Show that there exist $\alpha, \beta \in L$ such that $\alpha^2, \beta^2 \in \mathbb{Q}$ and such that $\{1, \alpha, \beta, \alpha\beta\}$ is a basis for L over $\mathbb{Q}$.

Describe, by drawing diagrams or otherwise, the lattice of subgroups of $G(L/\mathbb{Q})$, and the corresponding lattice of fields between L and $\mathbb{Q}$. **(15 marks)**

4 Let $f(t) = t^4 - 2 \in \mathbb{Q}[t]$ and let $L \subseteq \mathbb{C}$ be a splitting field for $f(t)$.

Write $\alpha = \sqrt[4]{2} > 0$.

(a) Show that $L = \mathbb{Q}(\alpha, i)$ and explain why L is normal. **(5 marks)**

(b) Find $[L : \mathbb{Q}]$, stating clearly any general result about field extensions that you use. **(7 marks)**

(c) Show that there is an element $\sigma \in G(L/\mathbb{Q})$ such that $\sigma(i) = i$ and $\sigma(\alpha) = i\alpha$, stating clearly any general result that you use.

You may assume without proof that there is also an element $\tau \in G(L/\mathbb{Q})$ such that $\tau(i) = -i$ and $\tau(\alpha) = \alpha$.

Show that every element of $G(L/\mathbb{Q})$ can be written as $\sigma^m\tau^n$ for $m, n \geqslant 0$. Include in your answer a complete list of the elements of $G(L/\mathbb{Q})$ with the effect of each element on i and α. **(10 marks)**

(d) To which abstract group is $G(L/\mathbb{Q})$ isomorphic? Justify your answer briefly. **(3 marks)**

5 Let $f(t) = t^3 - 3t - 1 \in \mathbb{Q}[t]$.

(a) Show that $f(t)$ is irreducible over $\mathbb{Q}$, stating clearly any general result which you use. **(3 marks)**

(b) Let $\xi = e^{\pi i/9}$ and write

$$\alpha = \xi + \xi^{-1}, \qquad \beta = -(\xi^2 + \xi^{-2}), \qquad \gamma = -(\xi^4 + \xi^{-4}).$$

Show that each of α, β, γ is a root of $f(t)$. **(6 marks)**

(c) Show that $\alpha^2 = 2 - \beta$, that $\beta^2 = 2 - \gamma$ and that $\gamma^2 = 2 - \alpha$.

Deduce that the splitting field for $f(t)$ is $\mathbb{Q}(\alpha)$. **(8 marks)**

(d) Determine the elements of $G(\mathbb{Q}(\alpha)/\mathbb{Q})$ and say which standard group $G(\mathbb{Q}(\alpha)/\mathbb{Q})$ is isomorphic to. **(8 marks)**

End of Question Paper