Data provided: Formulae sheet

The University Of Sheffield.

MAS445

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013-2014**

Mathematics (Numerical methods and vector spaces) **Two hours**

Marks will be awarded for your best FOUR answers

1 (i) Classify the following differential equations as either elliptic, parabolic or hyperbolic.

(a) $\dfrac{\partial u}{\partial p} + 3\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial p^2} = u$ ***(1 mark)***

(b) $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial y \partial x} + \dfrac{\partial u}{\partial y} + 3 = -\dfrac{\partial u^2}{\partial y^2} - 2u$ ***(1 mark)***

(c) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y \partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial x} = 5$ ***(1 mark)***

(ii) Discuss the benefits/drawbacks of the three systems of numerical solutions, i.e. implicit, explicit and Crank-Nicolson. ***(6 marks)***

(iii) Consider the following differential equation

$$3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial x^2} + 42u = 0,$$

By using the explicit difference scheme, solve the differential equation between $0 \le x \le 1, 0 \le y \le 1$ for

$$\Delta x = h = \frac{1}{3}, \qquad \Delta y = k = \frac{1}{2},$$

and the following boundary conditions

$$u(1, y) = 0, \qquad u(x, 1) = 1 - x^2,$$

under the assumption that the solution is symmetric across both the x and y axes, *i.e.* that $u(x, y) = u(x, -y) = u(-x, y)$. ***(15 marks)***

(iv) State the order of the error. ***(1 mark)***

2 (i) Discuss the advantages/disadvantages of LU factorisation over 'traditional' matrix inversion. *(2 marks)*

(ii) Determine the L and U matrices for the following system

$$A\mathbf{x} = b, \qquad A = \begin{bmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$

(5 marks)

(iii) Determine L^{-1} and U^{-1} *(4 marks)*

(iv) Hence find the column vector $\mathbf{x}$ *(3 marks)*

(v) Find the LU decomposition for the following tri-diagonal matrix

$$M = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

(6 marks)

(vi) Using your results from part (v), or otherwise, show M^{-1} to be

$$M^{-1} = \frac{1}{63}\begin{bmatrix} 31 & -30 & 28 & -24 & 16 \\ -15 & 45 & -42 & 36 & -24 \\ 7 & -21 & 49 & -42 & 28 \\ -3 & 9 & -21 & 45 & -30 \\ 1 & -3 & 7 & -15 & 31 \end{bmatrix}$$

(5 marks)

3 (i) Write down the three properties of cubic splines and why they are important. *(3 marks)*

(ii) Write down the values for σ_0 and σ_n under the assumption of

$$f''(x_0) = 0, \qquad f'(x_n) = 0.$$

(2 marks)

(iii) Using the conditions derived in (ii), determine the cubic spline between the following data points

x	0	$\pi/3$	$2\pi/3$	π
f(x)	1	$1/2$	$-1/2$	-1

(18 marks)

4 The set $(1,0,0) = a_1$, $(0,1,0) = a_2$, $(1,1,1) = a_3$ is in $\mathcal{R}^3$.

(i) Show that this set is independent but not orthogonal.

(6 marks)

(ii) Find c_i, d_i, $i = 1, 2, 3$, such that

$$f = (1,2,4) = \sum_{i=1}^{3} c_i a_i,$$
$$g = (-1,-1,2) = \sum_{i=1}^{3} d_i a_i.$$

(8 marks)

(iii) Show that

$$||f||^2 \neq \sum_{i=1}^{3} |c_i|^2,$$
$$||g||^2 \neq \sum_{i=1}^{3} |d_i|^2,$$
$$(f,g) \neq \sum_{i=1}^{3} c_i d_i.$$

(6 marks)

(iv) Explain briefly why the equality does not hold in part (iii) and thus the advantages of using an orthonormal basis.

(5 marks)

5 (i) Digital signals $\{f[n]\}$ $(n = 1, 2, 3)$ of length 3 are obtained by sampling a random signal $f(t)$ at intervals $T/2$, where $f(t)$ has autocorrelation function

$$R_f(\tau) = \sigma^2 (Sa(\pi\tau/T))^2$$

where $Sa(x) = \sin(x)/x$, and σ, τ and T are constant.

Write down the correlation matrix R, and use it to derive the K-L basis.

(20 marks)

(ii) Such signals are to be compressed using only two members of the K-L basis. Find them. What is the associated mean square error?

(5 marks)

6 (i) In $\mathcal{L}^2[0, 1]$ find constants a, b, c, d, e, f so that the vectors (i.e. functions)

$$\phi_1(t) = a, \qquad \phi_2(t) = b + ct, \qquad \phi_3(t) = d + et + ft^2$$

form an orthonormal set.

(17 marks)

(ii) Find the best approximation to the vector $f(t) = t^3$ using $\{\phi_i\}_{i=1}^3$.

(8 marks)

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\, t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$