The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

MAS451 Measure and Probability **2 hours**

Full marks may be obtained by complete answers to three questions. All answers will be marked, but credit will be given only for the best three answers. Total marks 99.

1 (i) (a) Let S be a set. List the properties that Σ must have for (S, Σ) to be a *measurable space*. ***(3 marks)***

(b) Explain what is meant by a *measure* on (S, Σ). ***(2 marks)***

(ii) Let α and β be positive real numbers, and m_1 and m_2 be measures on (S, Σ).

(a) Show that $\alpha m_1 + \beta m_2$ is a measure on (S, Σ), where for each $A \in \Sigma$,

$$(\alpha m_1 + \beta m_2)(A) = \alpha m_1(A) + \beta m_2(A).$$

(4 marks)

(b) Suppose that m_1 and m_2 are both probability measures. Give a condition on α and β so that $\alpha m_1 + \beta m_2$ is also a probability measure. ***(3 marks)***

(iii) (a) If $A, B, C \in \Sigma$ with $C \subset B \subset A$, deduce that

$$m(B \cap (A - C)) = m(B) - m(C).$$

(3 marks)

(b) Calculate the Lebesgue measure of the Borel subset E of $\mathbb{R}$ defined by

$$E = \left[0, \frac{1}{2}\right] \cap \left\{ [0,1] - \bigcup_{n=1}^{\infty} \left(\frac{1}{7^{n+1}}, \frac{1}{7^n} \right) \right\}.$$

(5 marks)

(iv) Let (S, Σ) be a measurable space and (f_n) be a sequence of bounded measurable functions from S to $\mathbb{R}$.

(a) Define $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ and explain why these functions are measurable. ***(5 marks)***

(b) Suppose that $\liminf_{n\to\infty} f_n = \limsup_{n\to\infty} f_n$ *almost everywhere*. What can you say about the existence and measurability of $\lim_{n\to\infty} f_n$? ***(4 marks)***

(c) Define a sequence of functions (f_n) from $\mathbb{R}$ to $\mathbb{R}$ by

$$f_n(x) = \begin{cases} \left(1 + \dfrac{x}{n}\right) & \text{if } x \in (0,1) \\ 0 & \text{if } x \notin (0,1). \end{cases}$$

Is each f_n bounded and measurable? Does $\lim_{n\to\infty} f_n(x)$ exist for all $x \in \mathbb{R}$, and if so, does the limit give rise to a measurable function? ***(4 marks)***

2 (i) Let (S, Σ) be a measurable space. Explain what is meant by both a *π–system*, and a *λ–system*. State Dynkin's $\pi - \lambda$ lemma. ***(6 marks)***

From now on let (S_1, Σ_1) and (S_2, Σ_2) be measure spaces. If $E \in \Sigma_1 \otimes \Sigma_2$ and $y \in S_2$, let E_y be the *y-slice of E* defined by

$$E_y = \{x \in S_1, (x, y) \in E\}.$$

You may freely use the facts that

$$(E^c)_y = (E_y)^c, \quad (E - F)_y = E_y - F_y, \quad \left(\bigcup_{n=1}^{\infty} E_n\right)_y = \bigcup_{n=1}^{\infty} (E_n)_y,$$

and that $E_y \in \Sigma_1$.

(ii) If $f : S_1 \times S_2 \to \mathbb{R}$ is measurable, show that $f_y : S_1 \to \mathbb{R}$ is also measurable, where $f_y(x) = f(x, y)$ for all $x \in S_1$. ***(4 marks)***

(iii) Let m_1 and m_2 be finite measures on (S_1, Σ_1) and (S_2, Σ_2), respectively, and for each $E \in \Sigma_1 \otimes \Sigma_2$, define $\psi_E : S_2 \to \mathbb{R}$ by

$$\psi_E(y) = m_1(E_y).$$

In order to define the product measure $(m_1 \times m_2)(E) = \displaystyle\int_{S_2} m_1(E_y) dm_2(y)$, we must first show that ψ_E is measurable. Define

$$\mathcal{S} = \{E \in \Sigma_1 \otimes \Sigma_2; \psi_E \text{ is measurable}\}.$$

(a) Show that $\mathcal{S}$ is a λ-system. ***(10 marks)***

(b) Show that for all $A \in \Sigma_1, B \in \Sigma_2$, the product set $A \times B \in \mathcal{S}$. ***(3 marks)***

(c) Show that $\{A \times B; A \in \Sigma_1, B \in \Sigma_2\}$ forms a π-system. ***(2 marks)***

(d) Deduce that ψ_E is measurable. ***(2 marks)***

(iv) (a) If $g : S_2 \to \mathbb{R}$ is measurable and $A \in \Sigma_1$, deduce that $\tilde{g}_A : S_1 \times S_2 \to \mathbb{R}$ is measurable, where $\tilde{g}_A(x, y) = \mathbf{1}_A(x) g(y)$, for all $x \in S_1, y \in S_2$. ***(2 marks)***

(b) Suppose that $f : S_1 \to \mathbb{R}$ is measurable. Use the result of (a) to show that $h : S_1 \times S_2 \to \mathbb{R}$ is measurable, where $h(x, y) = f(x) g(y)$, for all $x \in S_1, y \in S_2$. ***(4 marks)***

3 Throughout this question (S, Σ, m) is a measure space.

(i) State the *monotone convergence theorem*, and use it to prove *Fatou's lemma*: if (f_n) is a sequence of non-negative measurable functions then

$$\liminf_{n\to\infty} \int_S f_n dm \geq \int_S \liminf_{n\to\infty} f_n dm.$$

(8 marks)

(ii) Suppose that the sequence (f_n) of (i) converges pointwise to f. Deduce that

$$\limsup_{n\to\infty} \int_S f_n dm \geq \int_S f dm.$$

(3 marks)

(iii) Let $f : S \to \mathbb{R}$ be a non-negative measurable function and define a sequence (f_n) of functions by $f_n(x) = \min\{f(x), n\}$ for all $x \in S, n \in \mathbb{N}$. Deduce that

$$\lim_{n\to\infty} \int_S f_n dm = \int_S f dm.$$

(5 marks)

(iv) State the *dominated convergence theorem*, and use it to find

(a)

$$\lim_{n\to\infty} \int_0^\infty \frac{1 - e^{-nx}}{(1 + x^2)(1 - e^{-2nx})} dx.$$

(9 marks)

(b)

$$\lim_{n\to\infty} \int_0^1 \frac{1 + nx}{(1 + x)^n} dx.$$

(8 marks)

4 Throughout this question, $(\Omega, \mathcal{F}, P)$ is a probability space.

(i) Let (A_n) be a sequence of events in $\mathcal{F}$. Define the events $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$, and deduce that

$$P\left(\liminf_{n\to\infty} A_n^c\right) = 1 - P\left(\limsup_{n\to\infty} A_n\right)$$

(6 marks)

(ii) State both parts of the *Borel–Cantelli lemma*, and prove the part that does not require an independence assumption. ***(6 marks)***

(iii) Consider a sequence of (independent) rolls of a fair die. Deduce that the run 1412 appears infinitely often. ***(7 marks)***

(iv) Let (X_n) be a sequence of random variables which is such that for all $\epsilon > 0, P\left(\limsup_{n\to\infty}(|X_n| \geq \epsilon)\right) = 0$. What can you say about the asymptotic behaviour of X_n as $n \to \infty$? ***(6 marks)***

(v) The characteristic function of a random variable X is the mapping $\Phi_X : \mathbb{R} \to \mathbb{C}$ defined by $\Phi_X(u) = \mathbb{E}(e^{iuX})$, for all $u \in \mathbb{R}$.

(a) Show that the mapping $u \to \Phi_X(u)$ is continuous from $\mathbb{R}$ to $\mathbb{C}$. ***(3 marks)***

(b) Write down the characteristic functions for the following random variables, (I) $X \sim N(a, \sigma^2)$, (II) $X = a$ (a.s.), where $a \in \mathbb{R}, \sigma > 0$. ***(2 marks)***

(c) What can you say about the asymptotic behaviour of the sequence (X_n) of random variables, where $X_n \sim N(a, 1/n)$? ***(3 marks)***

End of Question Paper