The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **June 2015**

Probability and Probability Distributions **2 hours**

RESTRICTED OPEN BOOK EXAMINATION.
Candidates may bring to the examination lecture notes and associated lecture material (including set textbooks) plus a calculator that conforms to University regulations.
Candidates should attempt ***ALL*** *questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 80.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \frac{3}{4}(1-x^2) & -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the distribution function of X. ***(5 marks)***

(b) Find $E(X)$ and $\mathrm{Var}(X)$. ***(7 marks)***

(c) Let $Y = e^X$. Find the probability density function of Y. ***(5 marks)***

2 Two dice, A and B, are both six-sided. Dice A has faces labelled $1, 2, 2, 3, 3, 4$ and B has faces labelled $1, 3, 4, 5, 6, 8$. Dice A is rolled and then dice B. Assume that both dice are fair (so that each face comes up with probability $1/6$) and that the rolls are independent.

(a) Describe a suitable sample space to use for this experiment. ***(3 marks)***

(b) Find the probabilities of the following events.

(i) The sum of the numbers shown on the two dice is 4.

(ii) The two dice show the same number.

(iii) The number shown on dice A is strictly greater than that shown on dice B. ***(7 marks)***

(c) Give the mean and variance of the number shown on dice B. ***(6 marks)***

3 Let R be the region defined by $R = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1$, and let X and Y be random variables with joint probability density function given by

$$f_{X,Y}(x, y) = \begin{cases} k(x^2 + y^2) & (x, y) \in R \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of k. ***(4 marks)***

(b) Find $P(X \geq 1/2, Y \geq 1/2)$. ***(4 marks)***

(c) Find the marginal probability density function of X. ***(3 marks)***

(d) Find the conditional probability density function of Y given that $X = x$, assuming $-1 \leq x \leq 1$. ***(4 marks)***

(e) Find the conditional expectation of Y given that $X = x$, assuming $-1 \leq x \leq 1$. ***(4 marks)***

4 Each customer in a shop, independently of other customers, spends an amount which is given by a random variable with mean £12 and standard deviation £4. Let S be the total amount spent by the first 100 customers to visit the shop.

(a) Give the mean and standard deviation of S. ***(2 marks)***

(b) Use Chebyshev's inequality to give a lower bound for $P(1150 \leq S \leq 1250)$. ***(4 marks)***

(c) Use the Central Limit Theorem to give an approximate value for the same probability. (You may use the fact that for a $N(0,1)$ random variable Z, $P(Z \geq 1.25) = 0.1056$ to 4 decimal places, and you may assume that you do not need to use a continuity correction.) ***(4 marks)***

5 Two species of bird, A and B, are known to have the following probabilities for different numbers of eggs in their nests:

Species	Prob. of 1 egg	Prob. of 2 eggs	Prob. of 3 eggs	Prob. of 4 eggs
A	0.1	0.2	0.4	0.3
B	0.6	0.2	0.1	0.1

A single nest is observed, and the number of eggs, X, in it is counted. The prior probability that the nest is of species A is p.

(a) Let $p = 2/3$. For each of $x = 1, 2, 3, 4$, find the posterior probability that the nest is of species A. ***(9 marks)***

(b) If the observed number of eggs is 4, for what value of p are the posterior probabilities of species A and B equal? ***(4 marks)***

(c) Under what condition on p will the posterior probability of species A be higher than that of species B for all possible observations? ***(5 marks)***

End of Question Paper