Data provided: Formula Sheet

MAS156

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

MAS156 Mathematics (Electrical and Aerospace) **2 hours 30 minutes**

*Attempt **ALL** questions.*
Each question in Section A carries 2 or 3 marks and each question in Section B carries 8 marks.

Section A

A1 Give an example of a function f which has a minimum at $x = 0$ but which has $f''(0) = 0$.
Give an example of a function g which has $g'(2) = 0$ but which does not have a minimum or a maximum at $x = 2$. ***(2 marks)***

A2 Calculate $\dfrac{\partial^2 f}{\partial x \partial y}$ where $f(x, y) = y \sin(xy)$. ***(3 marks)***

A3 Calculate the angle between the vectors $(1, 2, 3)$ and $(4, 5, 6)$. ***(3 marks)***

A4 Find all complex numbers z that satisfy the equation $|z-1| = |z-j|$. ***(3 marks)***

A5 Evaluate the indefinite integral $\displaystyle\int \frac{x^2}{1+x}\, dx$. ***(3 marks)***

A6 Find the general solution of the differential equation $t\dfrac{dy}{dt} + 2y = \sqrt{t}$. *(3 marks)*

A7 Find the inverse Laplace transform of the function $\dfrac{1-2s}{s^2+6s+13}$. *(3 marks)*

A8 Find $\displaystyle\lim_{x\to\pi/2} \frac{x\sin 2x}{\pi - 2x}$. *(3 marks)*

A9 Evaluate the determinant of the matrix $M = \begin{pmatrix} 1 & -2 & 1 \\ 3 & 0 & 2 \\ -1 & -2 & 2 \end{pmatrix}$. *(3 marks)*

A10 Find the eigenvalues of the matrix $M = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$. *(3 marks)*

Section B

B1 Find the solutions of $z^4 = j$ and plot them on the Argand plane.
Find the solutions of $(w-1)^4 = j$.

B2 State, without proof or justification, the limits

$$\lim_{x\to 0} \frac{\sin(x)}{x} \quad \text{and} \quad \lim_{x\to 0} \frac{\cos(x)-1}{x}.$$

Give the definition of the derivative of a function f at the point $x = a$.
Use this to prove from first principles that if $f(x) = \sin(x)$ then $f'(a) = \cos(x)$.

B3 State the domain and range of each of the following functions: $f(x) = \cosh(x)$ and $g(x) = \text{arccosh}(x)$.
Suppose that $y = \text{arcsinh}(x)$ and $z = e^y$. Show that

$$z^2 - 2xz - 1 = 0.$$

Find z and hence deduce a formula for $\text{arcsinh}(x)$ in terms of x.

B4 Let $I(a) = \int \frac{\mathrm{d}x}{x^2 + 4x + a}$, where a is a constant.

Find the indefinite integral $I(a)$ when

(i) $a = 0$,

(ii) $a = 4$,

(iii) $a = 5$.

B5 Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = \sin 2t.$$

Show that as $t \to \infty$, y is approximately equal to $\alpha \sin(2t - \phi)$, and determine the values of α and ϕ.

B6 (i) Sketch the function

$$f(x) = e^{-x} \cos 10x, \quad x \geq 0.$$

Show from your sketch that the equation

$$e^{-x} \cos 10x = 2x, \quad x \geq 0$$

has a single solution $x_* \in (0, \pi/20)$.

(ii) Devise the Newton-Raphson iterative scheme to solve the equation

$$e^{-x} \cos 10x - 2x = 0, \quad x \geq 0.$$

Using $x_0 = 0.1$ as a first approximation, use this scheme to find the solution correct to four decimal places.

B7 (i) Find the relationship between α and β if the system of equations

$$x - 2y + z = 0$$
$$2x + y + z = 0$$
$$4x + \alpha y + \beta z = 0$$

has a non-trivial solution. Find the general solution when this relationship holds.

(ii) Find the values of α and β for which the equations

$$x - 2y + z = 0$$
$$2x + y + z = 5$$
$$4x + \alpha y + \beta z = 0$$

have infinitely many solutions.

End of Question Paper

Formula Sheet for MAS156

Trigonometry

$\cos^2\theta = (1+\cos 2\theta)/2$
$\sin^2\theta = (1-\cos 2\theta)/2$
$\sin 2\theta = 2\sin\theta\cos\theta$

Hyperbolic Functions

$\cosh^2\theta = (1+\cosh 2\theta)/2$
$\sinh^2\theta = -(1-\cosh 2\theta)/2$
$\sinh 2\theta = 2\sinh\theta\cosh\theta$

Binomial theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \binom{n}{r}x^r + \ldots$$

where $\displaystyle\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$.

If n is a positive integer then the series terminates and the result is true for all x, otherwise, the series is infinite and only converges for $|x| < 1$.

Function	**Derivative**
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$\frac{-1}{\sqrt{a^2-x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2+x^2}$
$\sinh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2+a^2}}$
$\cosh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2-a^2}}$
$\tanh^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2-x^2}$

Integration-by-Parts

$$\int uv'\,dx = uv - \int u'v\,dx$$

Substitution for a Rational Function of $\sin x$ and $\cos x$

If $t = \tan\left(\frac{x}{2}\right)$ then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $\frac{dx}{dt} = \frac{2}{1+t^2}$.

Taylor expansion of $f(x)$ about $x = a$

$$f(a) + (x-a)f^{(1)}(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \ldots$$

Newton-Raphson formula for the root of $f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Table of Laplace transforms

$f(t)$	$F(s) = \mathcal{L}(f(t))$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, 2, \ldots)$
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
$e^{at}f(t)$	$F(s-a)$ (shift theorem)
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$