The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Mathematics Core II **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

There are 60 marks available on the paper.

1 Let $A = (-2, 0)$ and $B = (6, 0)$. Describe the locus of all points $P = (x, y)$ such that $|BP| = 3|AP|$. ***(3 marks)***

2 Consider the quotient

$$\frac{\ln(\cos x)}{x^2}.$$

Using two applications of L'Hôpital's rule, or otherwise, find the limit of this function as $x \to 0$. Hence show that

$$\lim_{x\to 0} (\cos x)^{1/x^2} = \frac{1}{\sqrt{e}}.$$

(5 marks)

3 Write down the Taylor series of $z = e^x \sin y$ around $(0, 0)$ as far as the degree 2 terms. ***(4 marks)***

4 Determine the value of the parameter a for which the equations

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 1 \\ 2x & + & y & - & z & = & 1 \\ x & + & 2y & + & 4z & = & a \end{array}$$

have more than one solution. For this value of a, find all solutions to the system of equations. ***(6 marks)***

5 Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ and $p \times q$ matrices respectively.

(i) Under what conditions on m, n, p and q can we form the sum $A + B$?

(ii) Under what conditions on m, n, p and q can we form the product AB? If this condition holds, give an explicit formula for the ijth entry of AB in terms of the entries of A and B.

(iii) Give explicit examples of matrices A and B where AB exists but BA does not. ***(4 marks)***

6 (i) State the multiplication rule for determinants.

(ii) Recall that the *adjoint* matrix $\mathrm{adj}(A)$ of an $n \times n$ matrix A satisfies $A\,\mathrm{adj}(A) = \det A.I_n$. Express $\det(\mathrm{adj}(A))$ in terms of $\det A$.

(iii) Show that

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c).$$

(6 marks)

7 (i) Define the terms *eigenvector* and *eigenvalue* of an $n \times n$ matrix A.

(ii) Work out the eigenvectors and eigenvalues of

$$A = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}.$$

(iii) Label the two eigenvectors you computed in part (ii) by $\mathbf{u}$ and $\mathbf{v}$ (your choice of labelling does not matter). Write down a matrix M whose columns are $\mathbf{u}/|\mathbf{u}|$ and $\mathbf{v}/|\mathbf{v}|$. What property of A allows us to conclude that $M^T M = I$? Without doing explicit matrix multiplications, write down the 2×2 matrix $M^T A M$, justifying your answer.

(iv) Suppose that $\mathbf{v}$ is an eigenvector of M. By considering $\mathbf{v}^T M^T M \mathbf{v}$, show that the corresponding eigenvalue is ± 1. ***(11 marks)***

8 Recall that $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

(i) Write down the derivative of $\cosh x$.

(ii) Give the formula for the arclength l of the graph of a function $y = f(x)$ over an interval $[a, b]$, and calculate the length of the graph of the function $f(x) = \cosh x$ between $-\ln(2)$ and $\ln(2)$. ***(5 marks)***

9 Find the volume of the solid obtained by rotating the graph of $f(x) = \sin x$ about the x-axis in the region $0 \leq x \leq \pi$. ***(4 marks)***

10 Evaluate $\displaystyle\iint_T e^{x^2}\, dA$ where T is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$.

(6 marks)

11 If

$$(x, y) = \left(\frac{\sin u}{\cos v}, \frac{\sin v}{\cos u} \right),$$

show that the Jacobian is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = 1 - x^2 y^2.$$

Assuming that the region $\{(x, y) \ : \ 0 \le x \le 1, 0 \le y \le 1\}$ corresponds to the region $A = \{(u, v) \ : \ u > 0, \ v > 0, \ u + v < \pi/2\}$, deduce that

$$\int_0^1 \int_0^1 \frac{dx\, dy}{1 - x^2 y^2} = \pi^2/8.$$

(6 marks)

End of Question Paper