The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Vectors and Mechanics **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets. The total number of marks available is 60.

1 Let $OABC$ be a parallelogram where the midpoints of the lines OA, AB and BC are L, M and N, respectively. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

(i) Express $\overrightarrow{OL}$, $\overrightarrow{OM}$ and $\overrightarrow{ON}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

(ii) Express $\overrightarrow{LM} \cdot \overrightarrow{MN}$ in terms of $|\boldsymbol{a}|$ and $|\boldsymbol{c}|$. ***(3 marks)***

2 Relative to the origin O, points A and B have position vectors $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$, respectively.

Find

(i) the angle in radians between $\boldsymbol{a}$ and $\boldsymbol{b}$ correct to two decimal places;

(ii) the component of $\boldsymbol{b}$ along $\boldsymbol{a}$;

(iii) $\boldsymbol{a} \times \boldsymbol{b}$;

(iv) the area of triangle AOB correct to two decimal places. ***(6 marks)***

3 Simplify the following expressions

(i) $\boldsymbol{i} + (\boldsymbol{j} \times \boldsymbol{k})$;

(ii) $(\boldsymbol{i} \times \boldsymbol{k}) \cdot \boldsymbol{j}$. ***(2 marks)***

4 (i) Find a parametric vector equation of line L_1 which passes through a point with position vector $3\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$ and is in the direction of vector $-\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$.

(ii) Find a parametric vector equation of line L_2 which passes through points with position vectors $\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $6\boldsymbol{i} - 3\boldsymbol{j} + 11\boldsymbol{k}$.

(iii) Find out if L_1 and L_2 intersect and, if so, state their point of intersection. ***(6 marks)***

5 A point P with the position vector $\boldsymbol{p}$ lies on a line L with the parametric vector equation $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{c}$ such that $\boldsymbol{p} \cdot \boldsymbol{c} = 0$.

(i) Find λ for the position vector $\boldsymbol{p}$.

(ii) Assume that the vector equation of line L can also be written in the form $\boldsymbol{r} \times \boldsymbol{c} = \boldsymbol{d}$, where $\boldsymbol{d}$ is a constant vector. Hence show that

$$\boldsymbol{p} = \frac{1}{|\boldsymbol{c}|^2}(\boldsymbol{c} \times \boldsymbol{d}).$$

(3 marks)

6 A projectile is launched from the origin O with speed V at an angle θ above the horizontal. Ignore the effect of air resistance.

If the horizontal and vertical displacements of the projectile at time t are x and z respectively, write down the equations for x and z in terms of V, θ and the acceleration due to gravity g.

Assume an area of ground is completely flat and level with the x-axis. Show that a projectile launched from this ground will have a maximum range when $\theta = \pi/4$. Hence derive an expression for this maximum range in terms of V and g.

(4 marks)

7 (i) The position of a thrown stone is

$$\boldsymbol{r}(t) = (1.6 \ + 12\ t)\,\boldsymbol{i} + \left(15\ t - 4.9\ t^2\right)\boldsymbol{j},$$

where the units are m and s as appropriate.
Find its velocity and acceleration. ***(2 marks)***

(ii) A car is travelling east at 72 km h^{-1}. It rounds a curve and 5 seconds later it is travelling North at 72 km h^{-1}. Find the average acceleration of the car over the 5s. ***(5 marks)***

(iii) John's car has a mass of 2000 kg. It is at rest on the road because it runs out of fuel. John then pushes the car exerting a force of 300 N. How long does it take for the car to reach a velocity of 3 m s^{-1}? ***(4 marks)***

8 (i) One end of a light, taut inextensible string of length L is attached to a fixed point O and a particle P is attached to the other end of the string. The particle moves on a smooth track banked at an angle α to the horizontal. The particle moves in a horizontal circle with O at its centre, so that the string is horizontal. Draw a clear force diagram showing all the forces acting on the particle ***(3 marks)***

(ii) Two small beads are threaded on a vertical smooth circular wire. The beads are connected by a light, taut inextensible string which runs round the wire. Draw a clear diagram showing the forces on the two beads. ***(4 marks)***

(iii) An elastic string of length 1.5 m is stretched to 2 m. If its modulus of elasticity is 12 N, find the tension in the string. ***(3 marks)***

9 (i) A block of mass 4 kg lies on a rough plane banked at an angle of 30° to the horizontal. The coefficient of friction between the block and the plane is $\frac{3}{4}$. Show that in the absence of any external force the block does not slip down the plane if it is initially at rest. ***(3 marks)***

(ii) What is the smallest force parallel to the plane which can be applied to make the block just begin to move down the plane. ***(2 marks)***

(iii) What is the smallest force parallel to the plane which can be applied to make the block just begin to move up the plane. ***(2 marks)***

Note: consider acceleration due to gravity to be $g = 10$ m s^{-2} in Q. 9

10 For each of the following physical quantities, state whether they are vectors or scalars and give their dimensions: (a) Momentum; (b) Power; (c) Stiffness. ***(3 marks)***

11 A penny of mass 0.1 kg is placed on a horizontal turntable that is then rotated at a fixed rate of 90 revolutions per minute. The penny is placed on the table at a distance r from the axis of rotation. If the maximum frictional force between the turntable and the penny is 0.6 N, calculate the maximum distance r at which the penny would stay on the turntable at this rotation rate. ***(5 marks)***

End of Question Paper