Data provided: Formula Sheet

MAS152

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2015–2016

MAS152 Essential Mathematical Skills and Techniques

3 hours

Attempt ***ALL*** *questions.*

Each question in Section A carries 3 marks,
each question in Section B carries 8 marks.

All solutions should be justified in full. Calculators should be relied upon only for simple steps like basic arithmetic and plugging numbers into elementary functions.

Section A

A1 Let $f(x) = \dfrac{x}{x+4}$. Sketch the curve $y = f(x)$.

A2 Let $f(x) = e^{2x} - 1$. Find $f^{-1}(x)$ and state its domain and range.

A3 If $f(x, y) = 4x^2\sqrt{y} + 5\cos(xy)$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$.

A4 Evaluate $\lim_{x\to 0}\left(\dfrac{x\tanh x}{\sin 2x}\right)$ using l'Hôpital's Rule.

A5 Find all the complex numbers z for which $|z - 1 - i| = 1$ and $\mathrm{Re}(z) = \mathrm{Im}(z)$.

A6 If $\mathbf{a} = (7, -2, -5)$ and $\mathbf{b} = (5, 1, 3)$, evaluate $\mathbf{a} \cdot \mathbf{b}$. Find a non-zero vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

A7 Find the definite integral $\displaystyle\int_0^{\pi} (x+1)\sin\frac{x}{2}\,dx$.

A8 Find the indefinite integral $\displaystyle\int x(\cos x^2)^2\,dx$.

A9 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. Find A^T and B^T and hence show that $(AB)^T = B^T A^T$.

A10 Find the general solution of the differential equation

$$x\frac{dy}{dx} = \frac{e^x}{x^2} - 3y.$$

A11 Find A^{-1} for the matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$. Use this to solve the simultaneous equations

$$\begin{aligned} 2x + 4y &= 14 \\ x - 3y &= -8. \end{aligned}$$

A12 Find the general solution of the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0$$

such that $y = 0$ when $x = 0$.

Section B

B1 Find all the stationary points of $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2 - 1$ and show that the stationary points not at $(0, 0)$ are minima.

B2 By evaluating all the necessary derivatives of $y = \ln(1 + x^2)$, find the first 2 non-zero terms of the Maclaurin Series expansion of y. Show that this series can also be obtained from the Maclaurin Series of $y = \ln(1+x)$ given on the Formula Sheet.

B3 Find the modulus and principal argument of the complex numbers $z_1 = 1 + i$ and $z_2 = \sqrt{3} + i$. Hence find all complex numbers z that satisfy the equation $z^6 = \dfrac{z_1}{z_2}$ and plot them on an Argand diagram.

B4 The position vector of a particle, $\mathbf{r}(t)$, is given by

$$\mathbf{r}(t) = (2t^2, t^2 - 4t, 3t - 5).$$

(i) Find the velocity and acceleration vectors of the particle.

(ii) Find the unit vector in the direction of $\mathbf{r}$ at $t = 1$. Hence show that, in this direction, the component of the velocity is 4 times the component of the acceleration.

B5 Evaluate the definite integral

$$\int_1^2 \frac{2t^2 + 3t + 1}{t^3 + t}\, dt$$

writing your answer to 2 decimal places.

B6 Find the value of α for which the following system of equations has infinitely many solutions and then find those solutions.

$$\begin{aligned} 4x - y - z &= 2 \\ 2x + \alpha y + z &= 4 \\ x - 2y - 2z &= -3 \end{aligned}$$

For the case $\alpha = -2$, without solving the equations state how many solutions you would expect and write down the solution of the corresponding homogeneous system of equations.

B7 Let $A = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & -2 \\ -2 & 0 & 5 \end{bmatrix}$. Find all eigenvalues and eigenvectors of A.

B8 Using Laplace Transforms or otherwise, find the solution to the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin t$$

subject to the initial conditions $y = 0$ and $\dfrac{dy}{dt} = 0$ at $t = 0$.

End of Question Paper

MAS140/151/152/156 Formula Sheet

These results may be quoted without proof unless proofs are asked for in the questions.

Trigonometry

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \text{cosec}^2 x$

$2\sin^2 x = 1 - \cos 2x$

$2\cos^2 x = 1 + \cos 2x$

$2\sin x \cos x = \sin 2x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$a\cos x + b\sin x = R\cos(x - \alpha)$

where $R = \sqrt{a^2 + b^2}$,

$\cos\alpha = \frac{a}{R}$ and $\sin\alpha = \frac{b}{R}$

$2\cos x \cos y = \cos(x + y) + \cos(x - y)$

$2\sin x \sin y = \cos(x - y) - \cos(x + y)$

$2\sin x \cos y = \sin(x + y) + \sin(x - y)$

$\cos x = \dfrac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$

$\sin x = \dfrac{2\tan(x/2)}{1 + \tan^2(x/2)}$

$\tan x = \dfrac{2\tan(x/2)}{1 - \tan^2(x/2)}$

Hyperbolic Functions

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\tanh x = \dfrac{\sinh x}{\cosh x}$

$\coth x = \dfrac{\cosh x}{\sinh x}$

$\text{sech} x = \dfrac{1}{\cosh x}$

$\cosh^2 x - \sinh^2 x = 1$

$2\cosh^2 x = 1 + \cosh 2x$

$2\sinh^2 x = \cosh 2x - 1$

$2\sinh x \cosh x = \sinh 2x$

$\text{sech}^2 x = 1 - \tanh^2 x$

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$, all x

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $x \geq 1$

$\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$, $|x| < 1$

Series

Sum of an arithmetic series:

$$\frac{\text{first term} \ + \ \text{last term}}{2} \times (\text{number of terms})$$

Sum of a geometric series: $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{1 - x^n}{1 - x}$

Binomial theorem: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dbinom{n}{r} x^r + \ldots$

where $\dbinom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$

If n is a positive integer then the series terminates and the result is true for all x, otherwise, the series is infinite and only converges for $|x| < 1$.

$$\left.\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\
\exp x &= e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\end{aligned}\right\} \text{valid for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (-1 < x \leq 1)$$

Differentiation

Function	Derivative	Function	Derivative
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\text{cosec}^2 x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$	$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	$\cot^{-1} x$	$-\dfrac{1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$
$\tanh x$	$\dfrac{1}{\cosh^2 x}$	$\coth x$	$-\dfrac{1}{\sinh^2 x}$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2+1}}$	$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}\,,\ \lvert x\rvert > 1$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}\,,\ \lvert x\rvert < 1$		
$\coth^{-1} x$	$\dfrac{1}{1-x^2}\,,\ \lvert x\rvert > 1$		

Integration

In the following table the constants of integration have been omitted.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$\int \frac{dx}{x} = \ln|x|$$

$$\int e^x\,dx = e^x$$

$$\int a^x\,dx = \frac{a^x}{\ln a} \quad (a > 0,\ a \neq 1)$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int \mathrm{cosec}^2 x\,dx = -\cot x$$

$$\int \sinh x\,dx = \cosh x$$

$$\int \cosh x\,dx = \sinh x$$

$$\int \mathrm{sech}^2 x\,dx = \tanh x$$

$$\int \mathrm{cosech}^2 x\,dx = -\coth x$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} \quad (|x| < a)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} \quad (|x| > a)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| \quad \left(= \tanh^{-1}\frac{x}{a} \quad \text{if } |x| < a\right)$$

$$\int \mathrm{cosec}\,x\,dx = \ln\tan\left(\frac{x}{2}\right) \quad \text{or} \quad \ln(\mathrm{cosec}\,x - \cot x)$$

$$\int \sec x\,dx = \ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \quad \text{or} \quad \ln(\sec x + \tan x)$$

$$\int \mathrm{cosech}\,x\,dx = \ln\tanh\left(\frac{x}{2}\right)$$

Integration by parts

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

Variable substitution in definite integral

If $x = \varphi(t)$ is a monotonic function in the interval $[\alpha,\,\beta]$ and $a = \varphi(\alpha)$, $b = \varphi(\beta)$, then

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

Variable substitution for a rational function of sin x and cos x

Let $t = \tan\left(\frac{x}{2}\right)$ then $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$ and $\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$.

Table of Laplace transforms

Function $f(t)$	Laplace transform $F(s)$
t^n	$\dfrac{n!}{s^{n+1}}$ (for $n = 0, 1, 2, \ldots$)
e^{at}	$\dfrac{1}{s-a}$
$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$
$\sinh\omega t$	$\dfrac{\omega}{s^2-\omega^2}$
$\cosh\omega t$	$\dfrac{s}{s^2-\omega^2}$
$e^{at}f(t)$	$F(s-a)$ (shift theorem)
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$