The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Algebra **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets. There is a total of 60 marks.

1 (i) Let G be a group, and $x \in G$ a fixed element. Define the *conjugacy class* $\mathrm{conj}_G(x)$ and the *centraliser* $\mathrm{cent}_G(x)$. ***(2 marks)***

(ii) In the permutation group S_5, let $\alpha = (12)(345)$ and $\beta = (15243)$. Express $\beta\alpha\beta^{-1}$ as a product of disjoint cycles, by applying the permutation β to the entries in the expression for α. Note that you have just produced an element belonging to the conjugacy class $\mathrm{conj}_{S_5}(\alpha)$, and different from α itself. Determine how many elements $\mathrm{conj}_{S_5}(\alpha)$ contains, taking care not to count any twice. By considering a permutation of the entries of α that produces a different expression for the same permutation α, write down a non-identity element of $\mathrm{cent}_{S_5}(\alpha)$, different from α. What is the cardinality of $\mathrm{cent}_{S_5}(\alpha)$? ***(4 marks)***

2 Let G be a group. We define the *centre* of G to be

$$Z(G) := \{g \in G \mid gx = xg \ \forall x \in G\}.$$

Prove that $Z(G)$ is a normal subgroup of G, by applying the subgroup criterion to show that it is a subgroup, and considering left and right cosets to show that it is normal. ***(4 marks)***

3 Consider the map $\theta : \mathbb{Z} \to \mathrm{GL}_2(\mathbb{R})$ defined by $\theta(n) := J^n$, where $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Here $\mathbb{Z}$ is the group of integers under addition, and $\mathrm{GL}_2(\mathbb{R})$ is the group of invertible 2-by-2 real matrices, under multiplication. Prove that θ is a group homomorphism. What does the First Isomorphism Theorem tell us in this particular example? It is not enough just to state the general theorem. ***(4 marks)***

4 Consider the map $\theta : \mathbb{C} \to M_2(\mathbb{R})$ defined by $\theta(a+bi) := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a, b \in \mathbb{R}$. Prove that θ is a ring homomorphism. Give two independent justifications for the statement that it is not a ring isomorphism between $\mathbb{C}$ and $M_2(\mathbb{R})$. (You could try to show that it is not a bijection. You could also consider that any "structural" property of a ring is shared by any other ring isomorphic to it.) ***(5 marks)***

5 (i) Let R be a non-zero commutative ring. What does it mean to say that R is a *field*? ***(1 mark)***

(ii) Give a brief reason why $\mathbb{F}_{13} := \mathbb{Z}/13\mathbb{Z}$ is a field. What is the multiplicative inverse in $\mathbb{F}_{13}$ of $\overline{5}$? ***(2 marks)***

(iii) In the quotient ring $R := \dfrac{\mathbb{F}_{13}[x]}{\langle x^2+1 \rangle}$, calculate $[x-5][x-8]$, expressing your answer in as simple a form as possible. Is R a field? Is $\dfrac{\mathbb{F}_3[x]}{\langle x^2+1 \rangle}$ a field? (You might like to consider the analogy with part (ii), replacing $\mathbb{Z}$ by $\mathbb{F}_3[x]$ and 13 by x^2+1.) ***(4 marks)***

6 Consider the map $\theta : M_2(\mathbb{R}) \to \mathbb{R}$ defined by $\theta(A) := \det(A)$, the determinant of A. Is θ a ring homomorphism? Justify your answer. ***(2 marks)***

7 Let V be a vector space over a field F, and let $S = \{v_1, \ldots, v_m\}$ be a finite subset of V.

(i) What does it mean to say that S is linearly independent? ***(1 mark)***

(ii) What is meant by $\mathrm{Span}(S)$, the *span* of S? ***(1 mark)***

(iii) Consider the subset $S := \{\cos^2 x, \sin^2 x, \sin 2x\}$ of the $\mathbb{R}$-vector space $C(\mathbb{R})$ of continuous real-valued functions of a real variable. Prove that the constant function 1 belongs to $\mathrm{Span}(S)$. Prove also that S is linearly independent. (Hint: in a putative linear dependence relation, plug in carefully chosen values of x to show that all the coefficients have to be 0.) Is $\{\cos^2 x, \sin^2 x, \cos 2x\}$ linearly independent? ***(4 marks)***

8 By solving a set of linear equations, find a basis for the kernel of the linear map $\ell : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\ell(\mathbf{x}) := A\mathbf{x}$, where $A := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$. ***(2 marks)***

9 Suppose that for some field F, and some integer $n \geq 1$, $A, B \in M_n(F)$, with B invertible.

(i) Suppose that $v \in F^n$ is an eigenvector for A, with eigenvalue λ. Prove that Bv is an eigenvector for BAB^{-1}, with eigenvalue λ. (Just multiply the matrix by the vector and see what happens.) ***(1 mark)***

(ii) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let $f_A, f_B \in L(\mathbb{R}^3)$ be the linear operators $\mathbf{x} \mapsto A\mathbf{x}$, $\mathbf{x} \mapsto B\mathbf{x}$, respectively.

(a) Describe each of f_A and f_B as a rotation through a certain angle about a certain axis. ***(2 marks)***

(b) Starting from the characteristic polynomial $\det(xI - A)$, show that if $v \in \mathbb{R}^3$ is a (real) eigenvector for f_A, then v is unique, up to scalar multiples. Using (i), find an eigenvector for BAB^{-1}. How can we describe $f_{BAB^{-1}}$ geometrically? ***(4 marks)***

10 On the space $C^\infty(\mathbb{R}_{>0}, \mathbb{R})$ of real-valued functions on the domain $\mathbb{R}_{>0}$, with derivatives of all orders, consider the linear operator ℓ defined by $\ell(y)(x) := x^2 \dfrac{dy}{dx}$. By solving a differential equation, find an eigenvector for ℓ, with eigenvalue 2. ***(2 marks)***

11 Let V be a vector space over $\mathbb{R}$, with an inner product $\langle , \rangle$. Let $T \in L(V)$ be a linear operator (i.e. a linear map from V to V).

(i) What does it mean for T to be *self-adjoint* with respect to $\langle , \rangle$? ***(1 mark)***

(ii) Suppose that T is self-adjoint with respect to $\langle , \rangle$, and that v_1, v_2 are eigenvectors for T, with eigenvalues λ_1, λ_2 respectively. (So $Tv_1 = \lambda_1 v_1$ and $Tv_2 = \lambda_2 v_2$, with v_1, v_2 non-zero.) Prove that if $\lambda_1 \neq \lambda_2$ then v_1 and v_2 are orthogonal with respect to $\langle , \rangle$. ***(2 marks)***

(iii) Now let $V = \mathbb{R}^n$, with the standard dot product, i.e.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} := \mathbf{x}^t \mathbf{y} = \sum_{i=1}^{n} x_i y_i, \text{ for any } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Let $T \in L(\mathbb{R}^n)$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $A \in M_n(\mathbb{R})$ is a *symmetric* matrix. Prove that T is self-adjoint with respect to $\langle , \rangle$. ***(2 marks)***

(iv) Use the special case $n = 2$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, finding the eigenvalues and eigenvectors, to illustrate what you have proved in (ii). ***(3 marks)***

12 You may assume that $\langle f, g\rangle := \displaystyle\int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\, dx$ defines an inner product on the $\mathbb{R}$-vector space $\mathbb{R}[x]$ of polynomials in one variable with real coefficients.

(i) Show that 1 and x are orthogonal with respect to this inner product. Find the norms of 1, x and $3+2x$. (For the norm of x, you might like to consider a trigonometric substitution.) ***(5 marks)***

(ii) Explain why $\langle f, g\rangle := \displaystyle\int_{-1}^{1} x f(x)g(x)\, dx$ does not define an inner product on $\mathbb{R}[x]$. ***(2 marks)***

End of Question Paper