The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **2015–2016**

Analysis I **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) (a) Give the precise definition of what it means for a sequence (a_n) of real numbers to converge to a limit $l \in \mathbb{R}$. **(2 marks)**

(b) State the *Archimedean property* of the real numbers. **(1 mark)**

(c) Use (a) and (b) to prove carefully that the sequence whose nth term is $\dfrac{1}{5} - \dfrac{7}{\sqrt{n}}$ converges to a limit. **(4 marks)**

(ii) A sequence (x_n) is defined recursively by $x_1 = 3.5$, and for $n = 2, 3, \ldots$,

$$x_{n+1} = \frac{1}{7}(x_n^2 + 12).$$

(a) Prove by induction that $3 \le x_n \le 4$ for all $n \in \mathbb{N}$. **(3 marks)**

(b) By considering $x_{n+1} - x_n$, prove that (x_n) is monotonic decreasing. **(4 marks)**

(c) Explain why (x_n) converges, and use the algebra of limits and any other general results you know from the lectures to find $\lim\limits_{n\to\infty} x_n$. **(4 marks)**

(iii) (a) Give the definition of a Cauchy sequence of real numbers. **(1 mark)**

(b) Prove that every Cauchy sequence of real numbers is bounded. **(3 marks)**

(Hint: Aim for a bound of the form $K = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |a_{N+1}|\}$, where N is some fixed natural number depending on the sequence.)

(c) If (a_n) and (b_n) are sequences of real numbers wherein (a_n) is Cauchy and (b_n) converges to zero, prove that (a_nb_n) converges to zero. **(3 marks)**

2 (i) Derive the inequality $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$. You may use the triangle inequality without proof. **(4 marks)**

(ii) Give the precise definition of what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to have a limit l at a point $a \in \mathbb{R}$. What does it mean for f to be continuous at a? **(2 marks)**

(iii) With f as above, define $|f| : \mathbb{R} \to \mathbb{R}$ by $|f|(x) = |f(x)|$, for all $x \in \mathbb{R}$.

(a) If $\lim_{x \to a} f(x) = l$, show that $\lim_{x \to a} |f|(x) = |l|$. **(2 marks)**

(b) If f is continuous at a, show that $|f|$ is also continuous at a. **(1 mark)**

(c) Give an example of a function f that is differentiable at 0, whereas $|f|$ is not. **(1 mark)**

(iv) Consider the function $f(x) = x \cos(1/x)$ with domain $\mathbb{R} \setminus \{0\}$.

(a) Investigate the limiting behaviour of $f(x)$ as $x \to 0$. Hence show that f has a continuous extension $\tilde{f}$ to the whole of $\mathbb{R}$. **(3 marks)**

(b) Is it true that $\tilde{f}$ is differentiable at every point in $\mathbb{R}$? Give arguments to support your conclusion. **(5 marks)**

(v) (a) State Rolle's theorem. **(1 mark)**

(b) Prove the mean value theorem for a function f that is continuous on $[a, b]$ and differentiable on (a, b), i.e. show that there exists $c \in (a, b)$ so that

$$f(b) - f(a) = f'(c)(b - a).$$

Hint: Consider $g(x) = f(x) - \alpha(x - a)$, where $\alpha = \dfrac{f(b) - f(a)}{b - a}$. **(4 marks)**

(c) For f as in (b), if $f'(c) < 0$ for all $c \in (a, b)$, show that f is strictly monotonic decreasing. **(2 marks)**

3 (i) (a) Define what is meant by the *open ball* $B(\boldsymbol{a}, r)$ where $\boldsymbol{a} \in \mathbb{R}^k$, $r > 0$, and by an *open set* $U \subseteq \mathbb{R}^k$. **(2 marks)**

(b) State what is meant by a *continuous function* $f\colon \mathbb{R}^k \to \mathbb{R}^l$. **(1 mark)**

(c) Let $U \subseteq \mathbb{R}^l$. State what is meant by the *inverse image* $f^{-1}[U] \subseteq \mathbb{R}^k$, for a function $f\colon \mathbb{R}^k \to \mathbb{R}^l$. Prove that if f is a continuous function, and $U \subseteq \mathbb{R}^l$ is open, then the inverse image $f^{-1}[U] \subseteq \mathbb{R}^k$ is also open. **(5 marks)**

(ii) Let $a < b$. State without proof which of the following intervals are open subsets of $\mathbb{R}$:

(a) $[a, b]$;

(b) $[a, b)$;

(c) (a, ∞);

(d) $(-\infty, b]$.

(4 marks)

(iii) Which of the following subsets of $\mathbb{R}^2$ are open? Justify your answers. You may assume without proof that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y^2$ is continuous.

(a) $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$;

(b) $B = \{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$;

(c) $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

(6 marks)

4 (i) (a) Consider a sequence of functions $f_n\colon [a,b] \to \mathbb{R}$. State the definition of the sequence (f_n) converging *pointwise* to a function $f\colon [a,b] \to \mathbb{R}$. What is meant by the statement that the sequence (f_n) converges *uniformly* to f? **(2 marks)**

(b) If each function $f_n\colon [a,b] \to \mathbb{R}$ is continuous, and the sequence (f_n) converges uniformly to the function $f\colon [a,b] \to \mathbb{R}$, prove that f is continuous, and that

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

You may use the mean value theorem for integrals without proof. **(10 marks)**

(c) Find the pointwise limit of the sequence of functions $f_n\colon [0,1] \to \mathbb{R}$ defined by the formula $f_n(x) = x^n$. Why is the convergence not uniform? **(3 marks)**

(ii) (a) State the Weierstrass M-test on uniform convergence of series of functions. **(2 marks)**

(b) Let $r < 1$. Prove that the series

$$\sum_{k=0}^{\infty} x^n$$

converges uniformly when $x \in [-r, r]$. What is the limit? You may use standard results on geometric series without proof. **(3 marks)**

(c) Use part (i)(b) to show that

$$\log\left(\frac{1+r}{1-r}\right) = 2\left(r + \frac{r^3}{3} + \frac{r^5}{5} + \cdots\right).$$

(5 marks)

(iii) (a) State what is means for a function $f\colon I \to \mathbb{R}$ to be *uniformly continuous*, where I is some interval in $\mathbb{R}$. **(2 marks)**

(b) State a general result from which it follows directly that the function $g\colon [1,2] \to [1,2]$ defined by $g(x) = 1/x$ is uniformly continuous. **(2 marks)**

(c) Define a function $h\colon [1,\infty) \to [1,\infty)$ by the formula $h(t) = 1/t$. Let $1 \le x < y$. By using the mean value theorem over the interval $[x,y]$, show that the function h is uniformly continuous on $[1,\infty)$. **(3 marks)**

End of Question Paper