The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

MAS222 Differential Equations **2.5 hours**

Attempt ***ALL*** *questions. The allocation of marks is shown in brackets. Total marks 100.*

1 (i) Sketch the phase line for the following ordinary differential equation (ODE) for $-3\pi \leq u \leq 3\pi$.

$$\frac{du}{dx} = \sin(u).$$

State all the equilibrium points for $u \in \mathbb{R}$, and say which are stable and which are unstable. ***(3 marks)***

(ii) Consider the following system of ODEs

$$\frac{du}{dx} = u - v,$$
$$\frac{dv}{dx} = au + v,$$

where a is a real number. Determine the nature (e.g. node, saddle, spiral, centre etc.) and stability of the equilibrium point at (0,0) in the cases $a < -1$ and $a > 0$. ***(3 marks)***

Sketch the phase portraits for each case. ***(4 marks)***

(iii) Consider the following system of ODEs

$$\frac{dx}{dt} = y(x+1),$$
$$\frac{dy}{dt} = x(2 - x - y).$$

Find the equilibrium points. Determine their nature and stability. ***(8 marks)***

Sketch the nullclines. ***(2 marks)***

On a **separate diagram**, sketch the phase portrait for the system. ***(5 marks)***

2 (i) Verify that $y = x^{-2}$ is a solution to the equation $x^2y'' + 5xy' + 4y = 0$. Use reduction of order to find another (linearly independent) solution.

(6 marks)

(ii) Consider the following ODE

$$x^2y'' + xy' - y = 0. \tag{1}$$

Show that the normal form for Equation (1) is given by

$$4x^2u'' - 3u = 0. \tag{2}$$

(6 marks)

(iii) Show that $x = 0$ is a regular singular point of Equation (2). Find all Frobenius series solutions to Equation (2). ***(12 marks)***

(iv) Using parts (ii-iii), show that the general solution to Equation (1) is $y(x) = Ax^{-1} + Bx$. ***(1 mark)***

3 (i) We consider the following heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \ t > 0$$

with the following boundary conditions:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = u(\pi, t) = 0 \text{ for } t \geq 0. \qquad (3)$$

(a) Let $u(x,t) = T(t)X(x)$ be a separable solution of the equation. Show that $T(t)$ and $X(x)$ satisfy the following relations:

$$T'(t) = \alpha T(t), \quad X''(x) - \alpha X(x) = 0, \quad X'(0) = X(\pi) = 0,$$

where α is a constant. Explain why α is a constant. ***(4 marks)***

(b) Find the non-trivial solutions for $X(x)$. You may assume that the solution for $X(x)$ is trivial when $\alpha \geq 0$. ***(5 marks)***

(c) Find the solutions for $T(t)$ corresponding to the the above solutions for $X(x)$, and then write down the general solution for $u(x,t)$. ***(3 marks)***

(d) Making use of the above general solution, **explain briefly in words** how you may find the general solution for an inhomgeneous heat equation such as the following one:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-t}x, \quad 0 < x < \pi, \ t > 0$$

subject to the same boundary conditions given in Equation (3) above. ***(4 marks)***

(ii) Using the method of characteristics, find the solution for the following first order partial differential equation

$$\frac{\partial u}{\partial t} - 2t\frac{\partial u}{\partial x} = x, \quad u = u(x,t),$$

with $u(x,0) = e^{-2x}$. ***(9 marks)***

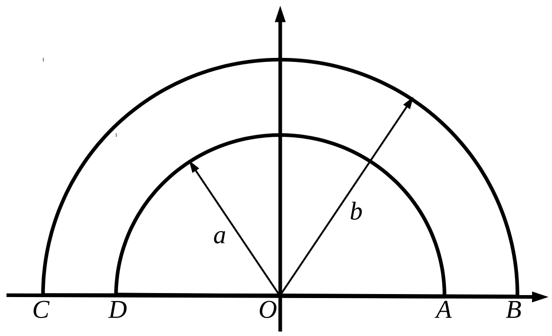

Figure 1: The half-ring solution domain for question 4(ii).

4 (i) Find a set of characteristic coordinates $\xi(x,t)$ and $\eta(x,t)$ for the following equation:

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial x \partial t} - 6x^2\frac{\partial^2 u}{\partial t^2} - t\frac{\partial u}{\partial t} = x^2.$$

Note: You do NOT need to use the characteristic coordinates to simplify the PDE. ***(5 marks)***

(ii) Let $u(r,\theta)$ be a function of r and θ, where r and θ are polar coordinates. $u(r,\theta)$ satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

in a half-ring, which is illustrated in Figure 1. The solution domain is defined by $a \le r \le b$ and $0 \le \theta \le \pi$, i.e., the radii of the inner and outer circles are, respectively, a and b.

(a) It is given that $u = 0$ on the inner circle $r = a$ as well as the two straight boundaries AB and CD. Find all the non-trivial separable solutions that satisfy these boundary conditions.

You may use the fact that the following problem has only the trivial solution when $\alpha \le 0$:

$$X''(x) + \alpha X(x) = 0, \quad X(0) = X(\pi) = 0,$$

where α is a constant. ***(13 marks)***

(b) Find the solution that also satisfies the condition $u = 1$ on the outer circle $r = b$.

You may use the following formula: for a function $f(\theta)$ defined in $\theta \in (0,\pi)$, its Fourier sine series is given by

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin(n\theta), \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin(n\theta)d\theta.$$

(7 marks)

End of Question Paper