The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Mathematics
(Computational and Numerical Methods) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) When applying the bisection method to find a root of a function, the tolerance ϵ is an upper bound on the actual error and is given by the following inequality

$$\epsilon \geq \frac{1}{2^n}(b-a), \tag{1}$$

where $n = 1, 2, 3, \ldots$ and (a, b) is the initial interval.

For $\epsilon = 10^{-3}$ and the interval $(0.655, 0.705)$ find the smallest possible value of n that satisfies inequality (1). Hence use this value of n to find an approximation to the root of function

$$f(x) = \exp\left(-x^2\right) - \sin(x) \tag{2}$$

in the interval $(0.655, 0.705)$. State your final answer to an accuracy of 3 decimal places. ***(9 marks)***

(ii) For the same $f(x)$ defined by (2) show that the Newton-Raphson method leads to a recurrence relation of the form

$$x_{n+1} = \frac{\exp(x_n^2)\,[x_n - \tan(x_n)] + (2x_n^2 + 1)\sec(x_n)}{\exp(x_n^2) + 2x_n \sec(x_n)}.$$

(4 marks)

(iii) Use the Gauss-Seidel iteration method to find an approximate solution to the following system of equations

$$\begin{aligned} x_1 + x_2 + 3x_3 &= -6 \\ 4x_1 - 2x_2 - x_3 &= 8 \\ 3x_1 - 5x_2 + x_3 &= 10. \end{aligned}$$

If necessary, first rearrange these equations to ensure convergence. Then starting with the initial column vector $\mathbf{x} = [1, 1, 1]^T$ compute four successive iterations, giving your final answer accurate to 4 decimal places. ***(12 marks)***

2 (i) It is required to find the interpolating polynomial, $P_3(x)$, through the points $(0, 4)$, $(1, 0)$, $(2, -6)$ and $(3, -8)$. Implementing the Lagrange interpolating formula calculate $L_0(x)$, $L_1(x)$, $L_2(x)$, $L_3(x)$ and hence find the interpolated values of $P_3(2.5)$, $P_3'(2.5)$ and $P_3''(2.5)$.

Hint: The Lagrange interpolation polynomial of least degree which passes through $(n + 1)$ points (x_i, y_i), $i = 0, 1, 2, \ldots, n$ is

$$P_n(x) = \sum_{i=0}^{n} L_i(x) y_i$$

where

$$L_i(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}$$

and $y_i = y(x_i)$.

(10 marks)

(ii) Assume the attenuation of sound in sea water is proportional to a power of the frequency. Using a least squares fit approximation find the power law associated with the following data, assuming the frequency measurements are error free.

Frequency (kHz)	0.5	1	2	5	10	20
Attenuation (dB/km)	0.02	0.07	0.14	0.33	0.92	3.20

In your calculation, if required, implement partial pivoting and give your final least squares fit parameters correct to an accuracy of 2 decimal places.

Hint: A polynomial of degree n can be expressed by the following sum,

$$P_n(x) = \sum_{j=0}^{n} a_j x^j.$$

In the least squares sense, a unique polynomial of degree n can be fitted to data points $(x_i, f(x_i))$, where $i = 0, 1, 2, \ldots, m$ and $m \geq n$. Assuming that the x_i values are free of errors, the normal equations used in the process of a least squares fit for a polynomial of degree n are

$$\sum_{i=0}^{m} \left(\sum_{j=0}^{n} a_j x_i^{j+k} \right) = \sum_{i=0}^{m} x_i^k f_i, \quad k = 0, 1, 2, \ldots, n.$$

(15 marks)

3 (i) Use the data points (x_0, f_0), (x_1, f_1) and (x_2, f_2) to derive Simpson's rule assuming $x_1 - x_0 = h$ and $x_2 - x_0 = 2h$, where $h > 0$ is a constant.

(10 marks)

(ii) Using the composite Simpson's rule evaluate

$$\int_{0.4}^{2.4} \left[\frac{5}{12}x^4 - \frac{1}{60}(x-1.5)^6\right] dx$$

to an accuracy of $\epsilon = 10^{-3}$. Give your final answer to an accuracy of 3 decimal places.

Hint: If a function $f(x)$ has four continuous derivatives on an interval (a, b) and this interval is divided into n subintervals, where n is an even positive integer, then the error bound for Simpson's method is given by

$$\left|E_n^S\right| \leq \frac{h^4}{180}(b-a)K,$$

where

$$h = \frac{b-a}{n}$$

and

$$K = \max_{a \leq x \leq b} \left|\frac{d^4 f(x)}{dx^4}\right|.$$

(15 marks)

4 (i) The first and second order derivatives of the function $y(x)$ can be approximated in finite difference form by

$$y_n' = \frac{y_{n+1} - y_{n-1}}{2h} \quad \text{and} \quad y_n'' = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2},$$

where y_n and $y_{n\pm1}$ denote $y(x_n)$ and $y(x_n \pm h)$, respectively. Use these relations with $h = 0.25$ to show that the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 3$$

with boundary conditions $y(1) = 1$ and $y'(1.75) = -2$ may be approximated by a system of linear algebraic equations in the form $\mathbf{Ay} = \mathbf{b}$. Determine the elements of matrix $\mathbf{A}$ and column vector $\mathbf{b}$ to an accuracy of 4 decimal places. **Do not attempt to solve these equations**. ***(10 marks)***

 Question 4 continued on next page

4 (continued)

(ii) A company manufactures 2 types of product, P_1 and P_2, which require 2 types of raw materials, A and B. Each unit of P_1 uses 1 kilogram of A and 3 kilograms of B, while each unit of P_2 uses 4 kilograms of A and 5 kilograms of B. The raw material available each day is 200 kilograms of A and 375 kilograms of B. It is expected that at least 1 unit of P_2 will be sold for every 4 units of P_1. The net profit per unit of P_1 is £ 40 and of P_2 is £ 30.

Formulate this into a linear programming problem and use graphical methods to determine the maximum possible daily profit. On the graph, clearly show the feasibility region and the line of constant revenue through the point of maximum daily profit. ***(15 marks)***

End of Question Paper