The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015-2016**

Mechanics and Fluids **2 hours**

Attempt all four questions. The allocation of marks is shown in brackets.

1 (i) A semi-circular laminar of radius R and uniform density σ lies in the upper (x, y)-plane, with its flat base centred on the origin. Find the position of its centre of mass in two ways:

(a) Split the semi-circle into horizontal strips of width δy and show that the mass of a strip is $2\sigma\sqrt{R^2 - y^2}\,\delta y$. Hence calculate the position of centre of mass.

(b) Change to polar coordinates and perform a double integral over the area.

(15 marks)

(ii) (a) State the rotational analogue of Newton's second law involving the moment of inertia.
Show that the moment of inertia of a uniform thin rod of length L and mass m about an axis through one end is

$$\frac{1}{3}mL^2\,.$$

(b) A beater, as used to strike a drum, is constructed of a thin rod of length 30 cm and mass 60 g, with a ball at one end that may be modelled as a point mass of 30 g. Calculate the moment of inertia for the beater about an axis through its end (the opposite end from the ball) in units kg m^2.
Starting from rest, a torque of 0.15 Nm is applied to the beater about this axis for 0.2 seconds before it strikes the surface of a drum. Calculate the velocity of the ball at the end of the beater at the instant it first makes contact with the drum.

(10 marks)

2 A particle of mass m is subject to a potential force $\boldsymbol{F} = -\boldsymbol{\nabla} V$ along a path $\boldsymbol{r}(t)$ from position A at time t_A to B at time t_B.

(i) The work done along the path is

$$W = \int_A^B \boldsymbol{F} \cdot \mathbf{d}\boldsymbol{r}\,.$$

Convert this path integral to a time integral. Using the first-order Taylor expansion for a small change in position for the potential V, show that $\mathrm{d}V/\mathrm{d}t = \boldsymbol{\nabla} V \cdot \dot{\boldsymbol{r}}$, where $\dot{\boldsymbol{r}}$ denotes $\mathbf{d}\boldsymbol{r}/\mathrm{d}t$, and hence that the work done along the path is equal to the difference in the potential at the end points.

(9 marks)

(ii) Consider $\boldsymbol{F} = y\,\boldsymbol{i} + (x+y^2)\boldsymbol{j}$. Calculate the work done using the line integral along the path from A to C formed by two straight line components AB and BC, where $A = (2, 2)$, $B = (2, 3)$, $C = (4, 3)$. Calculate a potential for the force and verify that the work done matches the change in the potential.

(16 marks)

3 (i) Let $\boldsymbol{\Omega}$ be a constant vector and $\boldsymbol{u} = \boldsymbol{\Omega} \times \boldsymbol{r}$. *Using suffix notation,* and the identity

$$\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}\,,$$

show that $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ and $\boldsymbol{\nabla} \times \boldsymbol{u} = 2\,\boldsymbol{\Omega}$. ***(9 marks)***

(ii) State Gauss's Theorem and show that it holds for the case $\boldsymbol{F} = z^3\,\boldsymbol{k}$ for a hemisphere defined by the limits $x^2 + y^2 + z^2 \leq a^2$ and $z \geq 0$. ***(16 marks)***

4 Euler's equation for the flow velocity $\boldsymbol{u}$ of an inviscid fluid is

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \boldsymbol{g},$$

where p is the pressure, ρ is the constant fluid density and $\boldsymbol{g}$ is a constant body force.

(i) If the flow is irrotational, show that this leads to the Bernoulli Integral

$$\frac{p}{\rho} = \boldsymbol{g} \cdot \boldsymbol{r} - \frac{\partial \phi}{\partial t} - \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u} + \beta\,,$$

where ϕ is to be defined and β is a constant (you may use the identities $(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla}(\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u})$ and $\boldsymbol{\nabla}(\boldsymbol{g} \cdot \boldsymbol{r}) = \boldsymbol{g}$). Find a simpler form valid when the flow is also steady and body forces may be neglected. ***(9 marks)***

(ii) The flow $\boldsymbol{u}$ of an incompressible fluid around a cylinder of radius a is given by

$$\boldsymbol{u} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta\,\hat{\boldsymbol{r}} - U\left[\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{a}{r}\right]\hat{\boldsymbol{\theta}}\,,$$

in cylindrical polar coordinates (r, θ, z), where U is a constant.

Verify that the appropriate boundary conditions are satisfied on $r = a$, and find the flow far from the origin.

Find the position of stagnation points in the flow.

Sketch the flow and explain in a few words why a net lift on the cylinder may be expected in this case. ***(16 marks)***

End of Question Paper