The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2015–2016**

Fluid Mechanics I — **2 hours**

Answer all four questions.

1 (i) Write down the Navier-Stokes (N-S) equation for an incompressible fluid of velocity $\mathbf{u}$ with constant density and give brief explanation of the terms in the N-S equations (you can ignore the body force for this part). Also, give an equation for the Reynolds number. ***(5 marks)***

(ii) Write down the continuity equation for an incompressible fluid of velocity $\mathbf{u}$ with constant density and explain its physical meaning. ***(2 marks)***

(iii) Let $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ be the vorticity. Show that

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nabla \left(\frac{1}{2}|\boldsymbol{u}|^2 \right) - \boldsymbol{u} \times \boldsymbol{\omega}.$$

Hence derive the time-dependent equation for the vorticity. ***(13 marks)***

(iv) Show by dimensional analysis that the equation

$$p + \frac{1}{2}\rho u^2 = H$$

is a possible relation between the pressure p and velocity u for frictionless flow along a streamline of a fluid of mass density ρ, and determine the dimensions of the constant H. ***(5 marks)***

2 Incompressible viscous fluid occupies the region between two infinite horizontal plates along $y = 0$ and $y = h$ and flows steadily in the x-direction with a velocity independent of z.

(i) Neglecting external body forces, write down the three components of the the Navier-Stokes equations, and show that the pressure gradient is constant and acts only in the x-direction.

(10 marks)

(ii) If the plates are fixed and $\dfrac{dp}{dx} = -P$, show that

$$\boldsymbol{v} = \frac{-P}{2\mu} y(y-h)\boldsymbol{e}_1,$$

where $\boldsymbol{e}_1$ is a unit vector in the x-direction.

(7 marks)

(iii) Calculate the volume flux per unit area across any fixed plane perpendicular to the motion.

(4 marks)

(iv) Calculate the drag per unit area exerted by the fluid on the plane $y = 0$.

(4 marks)

3 In the cylindrical polar coordinates (r, θ, z), the fluid velocity $\boldsymbol{v}$ in the *steady* flow of an incompressible fluid has components

$$\left(-\frac{1}{2}\alpha r, v(r), \alpha z\right)$$

where α is a positive constant. For $v = (v_r, v_\theta, v_z)$, you can use the following identities

$$(\boldsymbol{v}\cdot\nabla) = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z},$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},$$

and

$$\nabla\times\boldsymbol{v} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ v_r & rv_\theta & v_z. \end{vmatrix}$$

(i) Show if this fluid is incompressible or not.

(4 marks)

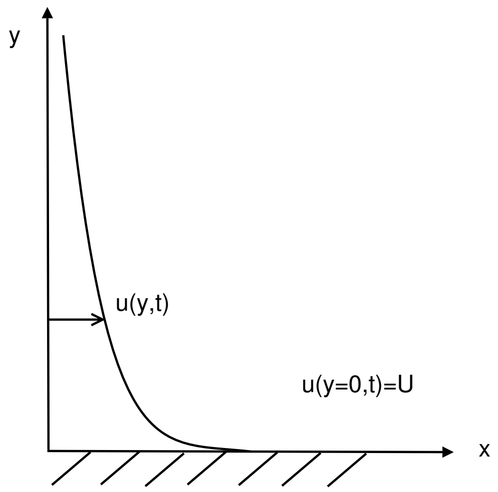
(ii) Calculate $\boldsymbol{\nabla}\times\boldsymbol{v}$ and state if this fluid is irrotational or not.

(6 marks)

(iii) Find the solution to the vorticty equation for this fluid.

(15 marks)

4 Consider that an *unsteady* viscous incompressible flow occupies the region $0 < y < \infty$, bounded by the rigid plate $y = 0$. The fluid is at rest when at $t = 0$, and the plane is jerked into motion in a direction along its length with velocity $(U, 0, 0)$ and thereafter moves at this constant velocity. Assume that the fluid is at rest initially with no pressure gradient. Let the velocity of the fluid $\boldsymbol{u} = (u(y,t), v, 0)$.



(i) Write the initial condition at $t = 0$ and two boundary conditions at $y = 0$ and $y \to \infty$ on the velocity.

(6 marks)

(ii) Show that the Navier-Stokes equation reduces to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \qquad (*)$$

(6 marks)

(iii) Show that (*) is unchanged by the transformation of variables.

$$y \Longrightarrow \alpha y,$$
$$t \Longrightarrow \alpha^2 t,$$

where α is a constant. ***(3 marks)***

(iv) Thus, find the velocity $\boldsymbol{u}$ by looking for the solution to (*) by using *similarity variable*

$$\eta = \frac{y}{2\sqrt{\nu t}}$$

where a factor 1/2 is included for convenience.

(10 marks)

End of Question Paper