The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015-2016**

Milestones in Applied Mathematics II **2 Hours**

Answer all four questions.

1 (i) Let A and B be operators on an inner product space. If A and B are self-adjoint operators, show that

$$[A, B]^* = [B, A],$$

(4 marks)

where $[A, B]^*$ denotes the adjoint operator of $[A, B]$.

(ii) In a region of space, a particle with mass m and with zero energy has a time-independent wave function

$$\psi(x) = Cx \exp(-x^2/L^2),$$

where C is a complex constant. Determine the potential energy $V(x)$ of the particle.

(8 marks)

(iii) The Hamiltonian of a quantum system is given by

$$H = \hbar\omega \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}.$$

At time $t = 0$, the state of the system is given by

$$\psi(0) = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}.$$

(a) Find the normalised stationary states.

(b) Find the state of the system at time t.

(13 marks)

2 A particle of mass m moves freely in the interval $[0, a]$ on the x-axis. You are *given* that its energy levels are

$$E_n = \frac{n^2\hbar^2\pi^2}{2ma^2}, \qquad n = 1, 2, \ldots,$$

and that the corresponding normalized wavefunction solutions of the *time-independent* Schrödinger equation are

$$\Psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right), & 0 < x < a; \\ 0, & \text{otherwise.} \end{cases}$$

(i) Write down $\psi_n(x,t)$, the normalized stationary state wavefunction solutions of the *time-dependent* Schrödinger equation. ***(2 marks)***

(ii) At $t = 0$, the wave-function is

$$\psi(x, 0) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

for $0 < x < a$, and $\psi(x, 0) = 0$ otherwise.

(a) Find the wave-function at a later time t.

(b) Find the probability that at time t the particle lies within the interval $[0, \frac{1}{2}a]$.

(8 marks)

(iii) At $t = 0$, the wave-function is

$$\psi(x, 0) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right) \left[1 + 2\cos\left(\frac{\pi x}{a}\right)\right]$$

for $0 < x < a$, and $\psi(x, 0) = 0$ otherwise.

(a) Find the wave-function at a later time t.

(b) Find the probability that at time t the particle lies within the interval $[0, \frac{1}{2}a]$. ***(15 marks)***

3 A beam of particles of energy $E > V > 0$ is incident *from the right* on the potential step

$$V(x) = \begin{cases} V & x > 0; \\ 0 & x < 0. \end{cases}$$

(i) Show that the time-independent Schrödinger equation is given as follows:

$$\frac{d^2\phi}{dx^2} + k^2\phi = 0, \quad \text{for } x < 0,$$

$$\frac{d^2\phi}{dx^2} + l^2\phi = 0, \quad \text{for } x > 0,$$

where $k^2 = \dfrac{2mE}{\hbar^2} > 0$ and $l^2 = \dfrac{2m(E-V)}{\hbar^2} > 0$.

(2 marks)

(ii) Solve the Schrödinger equations in part (i).

(9 marks)

(iii) Calculate the incident current, reflected, and transmitted currents.

(6 marks)

(iv) Calculate the reflection coefficient R and transmission coefficient T.

(4 marks)

(v) Calculate $R + T$ and state its physical meaning. ***(4 marks)***

4 Consider the Hamiltonian operator H for the one dimensional harmonic oscillator for a particle with mass m

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

where p and x are the momentum and position operator respectively, and ω is a constant. The operator A is defined as

$$A = \sqrt{\frac{m\omega}{2\hbar}}\,x + i\frac{p}{\sqrt{2m\hbar\omega}}.$$

(i) Find the adjoint operator A^* and calculate

$$[A, A^*].$$

(5 marks)

(ii) Show that

$$H = \hbar\omega\left(A^*A + \frac{1}{2}\right).$$

(6 marks)

(iii) Show that

$$[H, A] = -\hbar\omega A,$$

and

$$[H, A^*] = \hbar\omega A^*.$$

(4 marks)

(iv) Let the eigenstate of H be u_n. By applying the operators $[H, A]$ and $[H, A^*]$ to u_n, find the physical interpretation of of Au_n and A^*u_n and hence show that the allowed energies are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega,$$

where $n = 0, 1, 2.....$

(10 marks)

End of Question Paper