The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Spring Semester 2015–2016**

**Mathematical Methods** **2 hours**

*Marks will be awarded for your best **FOUR** answers. The marks awarded to each question or section of question are shown in italics.*

**1** The Fourier transform, $\hat{f}(k)$, of a function $f(x)$ is defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx.$$

(a) Write down the inverse Fourier transform, and use it to derive Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\hat{f}(k)\right|^2 dk.$$ ***(9 marks)***

(b) The function $f(x)$ is defined by

$$f(x) = \begin{cases} x & |x| \leqslant 1 \\ 0 & |x| > 1. \end{cases}$$

Find $\hat{f}(k)$, and use Parseval's theorem to deduce that

$$\int_{0}^{\infty} \left(\frac{\sin k - k\cos k}{k^2}\right)^2 dk = \frac{\pi}{6}\,.$$ ***(16 marks)***

**2** The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

(a) Find the Laplace transform of $\sin \omega t$ for $\operatorname{Re} s > 0$, where $\omega$ is a real constant. ***(5 marks)***

(b) The function $f(t)$ is defined by

$$f(t) = \begin{cases} 1-t & 0 \leqslant t \leqslant 1 \\ 0 & t > 1. \end{cases}$$

Find the Laplace transform of $f(t)$. ***(5 marks)***

(c) $F(t)$ is defined for $t > 0$ by

$$F(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

Find $F(t)$ for $t > 0$ if $f(t)$ is as defined in part (b), and $g(t) = \sin \omega t$ where $\omega$ is a real constant. ***(11 marks)***

(d) Given that for $t > 0$

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{f(t)\}\, \mathcal{L}\{g(t)\},$$

use the results of parts (a), (b) and (c) to deduce that if $\omega$ is a real constant then the inverse Laplace transform of

$$\frac{\omega(s + e^{-s} - 1)}{s^2(s^2+\omega^2)}$$

is

$$-\frac{1}{\omega}\cos \omega t + \frac{1}{\omega^2}\sin \omega t - \frac{H(t-1)}{\omega^2}\sin \omega(t-1) + \frac{H(1-t)}{\omega}(1-t) \quad \text{for } t > 0,$$

where $H$ is the Heaviside step function. ***(4 marks)***

**3** The function $y(x)$ satisfies the ordinary differential equation

$$x^2y'' - 2y = \ln(1+x) \qquad 0 \leqslant x \leqslant 1, \tag{1}$$

with the boundary conditions

$$y \text{ finite as } x \to 0$$

$$y = 0 \text{ at } x = 1.$$

(a) By trying $y = x^n$, find the independent solutions of

$$x^2y'' - 2y = 0.$$ ***(3 marks)***

(b) Given that Green's function $G(x;\xi)$ for the boundary-value problem given at the beginning of the question is continuous at $x = \xi$, and that $\partial G/\partial x$ has a discontinuity of size $1/\xi^2$ at $x = \xi$, show that

$$G(x;\xi) = \begin{cases} \frac{1}{3}\left(1 - \frac{1}{\xi^3}\right)x^2 & 0 \leqslant x < \xi, \\ \frac{1}{3}\left(x^2 - \frac{1}{x}\right) & \xi < x \leqslant 1. \end{cases}$$ ***(14 marks)***

(c) Use Green's function to write down the solution to equation (1) and the boundary conditions given at the beginning of the question (do NOT attempt the $\xi$ integrals). ***(3 marks)***

Use this to show that

$$y'(1) = 2\ln 2 - 1.$$ ***(5 marks)***

**4** Consider the equation

$$(1-\epsilon)x^2 + 4x + 4 = 0, \tag{2}$$

where $\epsilon$ is a constant satisfying $0 < \epsilon \ll 1$.

(a) The solution to equation (2) can be written as

$$x = x_0 + \epsilon^{1/2}x_1 + \epsilon x_2 + \epsilon^{3/2}x_3 + \epsilon^2 x_4 + \epsilon^{5/2}x_5 + \cdots,$$

where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\epsilon \to 0$.

Use this expression to derive the two solutions to equation (2), correct to order $\epsilon^2$ as $\epsilon \to 0$. ***(20 marks)***

(b) Find the exact solutions of (2), and show that their expansions agree with your results from part (a). ***(5 marks)***

**5** The Laplace transform, $\tilde{f}(s)$, of a function $f(t)$ is defined by

$$\tilde{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt,$$

where $s$ is real and positive.

(a) By integrating by parts, show that if $f$ is $n$ times differentiable then

$$\tilde{f}(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \cdots + \frac{f^{(n-1)}(0)}{s^n} + R_n(s), \qquad (3)$$

where

$$R_n(s) = \frac{1}{s^n} \int_0^\infty e^{-st} f^{(n)}(t)\, dt.$$

***(8 marks)***

$\left[\text{ You may assume that } \lim_{t\to\infty} \left\{e^{-st} f^{(m)}(t)\right\} = 0 \text{ for } m = 0, 1, \ldots, n-1. \right]$

(b) Use equation (3) to expand $\tilde{f}(s)$ for $f(t) = \ln(1 + 2t)$. ***(11 marks)***

Show that in this case

$$|R_n(s)| < \frac{2^n (n-1)!}{s^{n+1}},$$

and, hence, that the expansion is an asymptotic expansion as $s \to \infty$.

***(6 marks)***

## End of Question Paper