The University Of Sheffield.

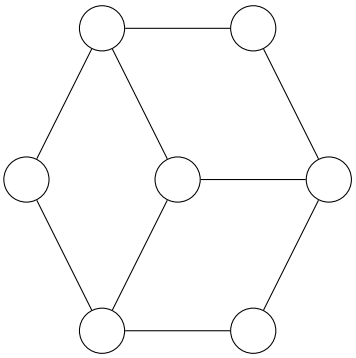
SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2015–2016

Graph Theory 2 hours 30 minutes

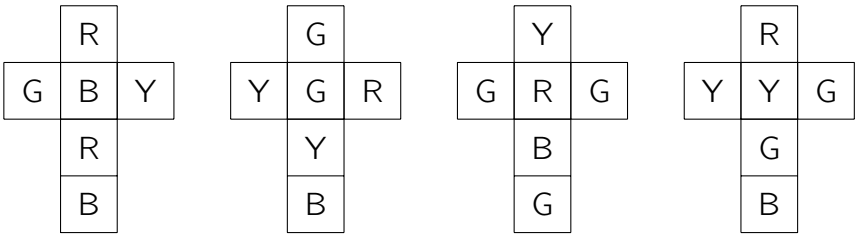
*Answer FOUR questions. You are advised NOT to answer more than four questions: if you do, only your best four will be counted.*

**1** (i) Explain why any alkane $C_nH_{2n+2}$ is a tree. How many isomers does $C_6H_{14}$ have? Draw the structure of the carbon atoms in each isomer. ***(5 marks)***

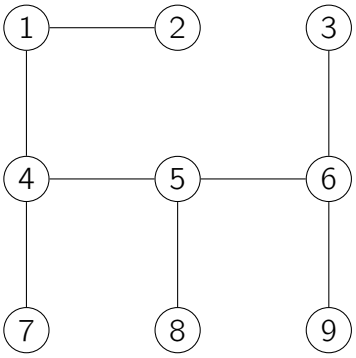
(ii) Consider the graph $G$ given below. Is $G$ Eulerian? Is $G$ Hamiltonian? Is $G$ bipartite? Justify your answers. ***(6 marks)***



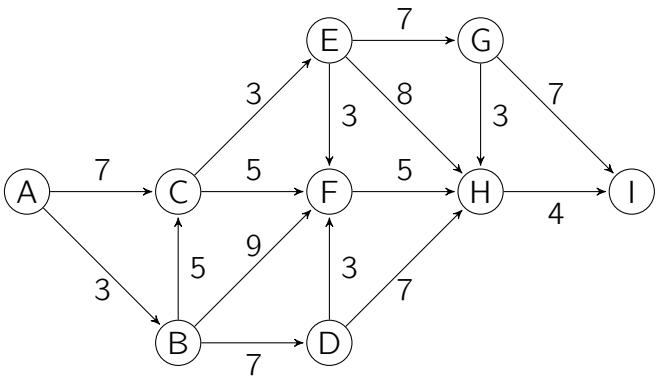
(iii) Prove that the following set of instant insanity cubes have no solution. ***(9 marks)***




(iv) Give the Prüfer code for the labelled tree $T$ below. ***(5 marks)***



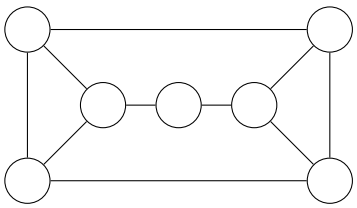
2 Consider the following directed, weighted graph:




(i) What is the length $s$ of the shortest path from $A$ to $I$? For which edges $e$ will shortening $e$ by 0.1 change $s$? For which edges $e$ will making $e$ longer by 0.1 change $s$? ***(8 marks)***

(ii) What is the length $\ell$ of the longest path from $A$ to $I$? For which edges $e$ will shortening $e$ by 0.1 change $\ell$? For which edges $e$ will making $e$ longer by 0.1 change $\ell$? ***(8 marks)***

(iii) The graph $\Gamma$ is shown below. Find the chromatic number and the chromatic index of $\Gamma$. ***(5 marks)***



(iv) A tree $T$ has one vertex $v$ of degree 4, and another vertex $w$ of degree 3. Prove that $T$ has at least 5 leaves. ***(4 marks)***

**3** (i) Weights are given for edges between 7 vertices, labelled $A - G$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $A$ | | | | | | |
| 11 | $B$ | | | | | |
| 17 | 9 | $C$ | | | | |
| 17 | 12 | 14 | $D$ | | | |
| 11 | 17 | 15 | 10 | $E$ | | |
| 16 | 9 | 9 | 10 | 8 | $F$ | |
| 20 | 10 | 21 | 19 | 8 | 12 | $G$ |

Find a minimal weight spanning tree. What is the total weight of this spanning tree? ***(5 marks)***

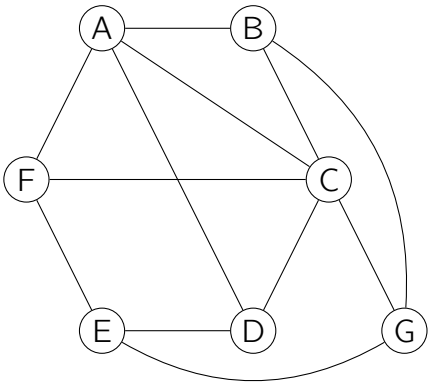
(ii) In total, how many spanning trees have the same minimum weight? ***(4 marks)***

(iii) Now, suppose the vertices represent towns, and the weights represent the cost of traveling between towns. A traveling salesperson lives in an 8th town, $H$. The cost of traveling from $H$ to any town other is 25. The traveling salesperson wants to start at $H$, travel to every town $A - G$ exactly once, and then return to $H$, as cheaply as possible. Using your result from the previous part, give a lower bound on the cost of the traveling salesperson's trip. Is this lower bound attainable? Explain. ***(5 marks)***

(iv) Using the nearest neighbour heuristic, starting at $H$ and traveling first to $G$, give an upper bound on the cost of the cheapest trip for the traveling salesperson. ***(3 marks)***

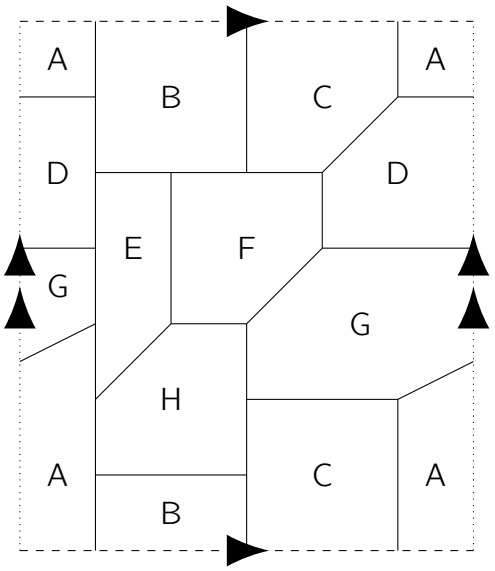
(v) Draw the Petersen graph $P$, and prove that $P$ is not Hamiltonian. (Hint: Suppose that $P$ is Hamiltonian, and consider the edges *not* in the Hamiltonian cycle) ***(8 marks)***

**4** (i) State Kuratowski's theorem, and use it to show that the graph $G$ below is not planar. Draw $G$ on the projective plane without edges crossing. Your drawing should use the labelling of the vertices given. ***(10 marks)***



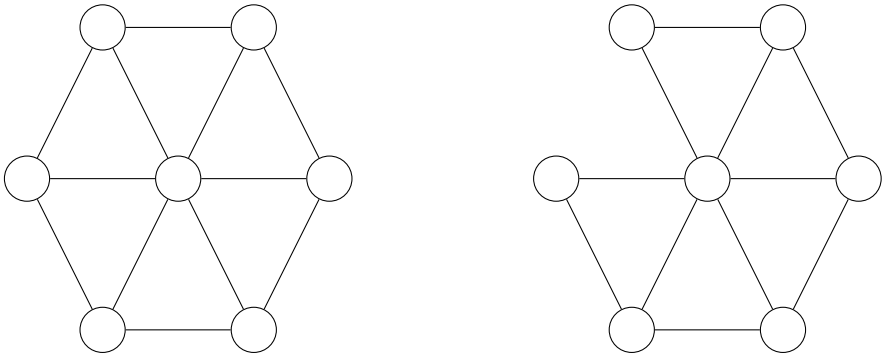

(ii) Define the Euler characteristic $\chi(S)$ of a closed, compact surface $S$ and prove it is well defined. Use your drawing of $G$ from Part (i) to calculate the Euler characteristic of the projective plane. ***(11 marks)***

(iii) Consider the graph $\Gamma$ drawn below on the torus, with its faces labeled A through H. Give a colouring of the faces of $\Gamma$ with four colours so that faces meeting along an edge have different colours. Prove that no such colouring is possible with only three colours. ***(4 marks)***



**5** Recall that the wheel graph $W_n$ consists of a copy of the cycle graph $C_{n-1}$, together with a central vertex $v$ adjacent to every other vertex. Let $BW_n$ denote the "broken wheel" graph, which is obtained from the wheel graph $W_n$ by removing one edge from the outer cycle. We have drawn $W_7$ and $BW_7$ below:

$W_7$ $BW_7$



(i) Prove, directly from the definition, that the chromatic polynomials of $W_n$ and $C_n$ satisfy the identity:

$$P_{W_n}(k) = kP_{C_{n-1}}(k-1)$$

***(5 marks)***

(ii) Prove that

$$P_{BW_n}(k) = k(k-1)(k-2)^{n-2}$$

***(5 marks)***

(iii) Prove that

$$P_{W_n}(k) = P_{BW_n}(k) - P_{W_{n-1}}(k)$$

Using this, the previous part, and induction, prove that

$$P_{W_n} = k(k-2)\left[(k-2)^{n-2} + (-1)^{n-1}\right]$$

***(10 marks)***

(iv) A graph $G$ has chromatic polynomial $P_G(k) = k^4 - 4k^3 + 5k^2 - 2k$. How many vertices and edges does $G$ have? Is $G$ bipartite? Justify your answers.

***(5 marks)***

## End of Question Paper