The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Knots and Surfaces **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.
Strings, pipe cleaners, shoe laces or similar aids for making knots may be used.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 State whether the following are true or false. *Carefully* justify your answer with a proof or counterexample as necessary. Most of the marks are awarded for the justification.

(i) Two different knots can project to the same knot universe.

(ii) Two different links can project to the same link diagram.

(iii) Suppose that α is a knot invariant. If k and k' are knot diagrams such that $\alpha(k) = \alpha(k')$ then k and k' are Reidemeister equivalent.

(iv) Suppose that l and l' are link diagrams such that for *every* link invariant β we have $\beta(l) = \beta(l')$ then l and l' are Reidemeister equivalent.

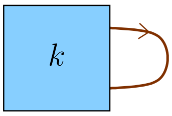
(v) The writhe is invariant under the first Reidemeister move.

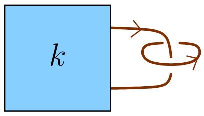
(vi) The writhe is invariant under the second Reidemeister move.

(25 marks)

2 (i) Explain briefly what it means to say that two link diagrams are Reidemeister equivalent. ***(4 marks)***

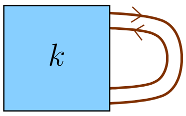
(ii) Show that if k' is obtained from k by adding a single component to k negatively linking one strand as illustrated then their Jones polynomials are related by the equation $f[k'] = -(A^{10} + A^2)f[k]$.

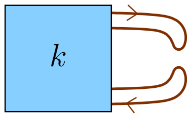
$k =$ 

, $k' =$ 

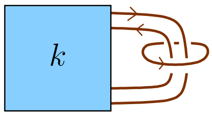
(5 marks)

(iii) Consider a link k and pick two oppositely oriented strands from it. Let $\mathrm{Cl}(k)$ be obtained from k by cutting the two strands and rejoining them as shown.

$k =$ 

, $\mathrm{Cl}(k) =$ 

Show that if k'' is obtained from k by adding a single component to k linking the two selected strands as illustrated

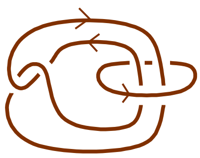
$k'' =$ 

then

$$f[k''] = -(A^6 + A^6)f[k] - (A^4 - 2 + A^{-4})f[\mathrm{Cl}(k)].$$

(10 marks)

(iv) Use the above results to calculate the Jones polynomial of the Whitehead link:



Is this link amphicheiral? Justify your answer. ***(6 marks)***

3 (i) (a) What is meant by a compact connected surface? ***(4 marks)***

(b) List the standard surfaces together with the standard surface words and state the classification of compact connected surfaces. ***(8 marks)***

(ii) (a) Write down the word operation called the *orientable handle move*. Draw a series of pictures to show that the associated surfaces are homeomorphic. ***(4 marks)***

(b) Use word operations to show that the surface with word $WabUa^{-1}b^{-1}$ is homeomorphic to the surface with word $UWaba^{-1}b^{-1}$, where U and W are strings of letters. ***(4 marks)***

(c) For each $n \geq 1$, use the above to find the genus of the surface with word $a_1a_2 \dots a_n a_1^{-1} \dots a_n^{-1}$. ***(5 marks)***

4 (i) (a) State the inclusion/exclusion principle for the Euler characteristic. Give the Euler characteristic of the disc, the circle and the Klein bottle. ***(6 marks)***

(b) A surface Σ is formed from two Klein bottles K_1 and K_2 by first making n holes in each of the Klein bottles and then joining the boundary circles together in pairs. If Σ is connected, which standard surface is it homeomorphic to? ***(6 marks)***

(ii) (a) Give a formula for the Euler characteristic of a covering pattern of a compact surface, explaining any symbol used. ***(2 marks)***

(b) Nine squares are joined together to form a surface so that four squares meet at each vertex. Use the above formula to find what possible surface(s) could have been formed. ***(5 marks)***

(c) You use n triangles to form a projective plane so that exactly t distinct triangles meet at each vertex. Find and prove all values of n and t for which this is possible. ***(6 marks)***

End of Question Paper