The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

MAS346 Groups and Symmetry **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Define the set Isom_n of isometries of $\mathbf{R}^n$ and show that it is closed under composition and taking inverses. ***(7 marks)***

(ii) Define $\psi : \mathrm{Isom}_n \to O_n$ and show that it is a homomorphism. ***(3 marks)***

(iii) The kernel of ψ is given by the translations $T_a : x \mapsto x + a$ for $a \in \mathbf{R}^n$. Prove that for any $f \in \mathrm{Isom}_n$ we have $fT_af^{-1} = T_{\psi(f)a}$. ***(4 marks)***

(iv) This part concerns plane isometries, i.e. $n = 2$.

(a) Define the elements R_θ, S_θ of O_2. ***(2 marks)***

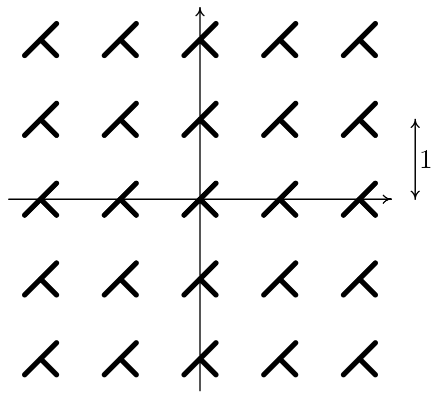
(b) Give an explicit formula for $R_{\theta,a}$, the rotation by angle θ about $a \in \mathbf{R}^2$, and determine $\psi(R_{\theta,a})$. ***(2 marks)***

(c) Let A be an element of O_2 with $\det(A) = -1$. Prove that $A = S_\theta$ for some θ. ***(4 marks)***

(d) Let $f \in \mathrm{Isom}_2$ with $\det(\psi(f)) = -1$. Show that f^2 is a translation. ***(3 marks)***

2 (i) For any subgroup $H \leq \mathrm{Isom}_2$ we defined its point group $\psi(H) \leq O_2$ and translation subgroup $\mathrm{Trans}(H) \leq \mathbf{R}^2$. Explain which properties $\psi(H)$ and $\mathrm{Trans}(H)$ need to satisfy for H to be a wallpaper group. ***(3 marks)***

(ii) Let G be the isometry group of the infinite wallpaper pattern, a portion of which is illustrated below. (A copy of the diagram on white paper is provided; if you wish, you may write on it and hand it in with your answer.)



(a) Describe geometrically *all* the translations, reflections and rotations (if any) in G. State clearly the vectors of any translations, lines of any reflections, and the centres and angles of any rotations. Specify one more element of G that is not a translation, rotation or reflection. ***(8 marks)***

(b) Find a list of three isometries that generate G. Justify your answer. ***(10 marks)***

(c) Find n and θ such that $\psi(G)$ is equal to $R_\theta D_n R_\theta^{-1}$. Justify your answer. ***(4 marks)***

3 (i) (a) Give the definition of the action of a group G on a set X. ***(3 marks)***

(b) Given a group action explain how to define the corresponding map $\phi : G \to S(X)$ and prove that it is a homomorphism taking values in $S(X)$. ***(5 marks)***

(ii) Let a group G act on a set X.

(a) Show that this induces a group action of G on the set of subsets of X by the rule $g * N := gN = \{gn | n \in N\}$ for $N \subseteq X$. ***(4 marks)***

(b) If G is a group with more than one element acting on itself by left multiplication then show that $g * N$ does not induce an action on the set of subgroups of G. ***(3 marks)***

(iii) Prove that the symmetry group of a regular tetrahedron centered at the origin is isomorphic to S_4. ***(6 marks)***

(iv) Describe a subset of $\mathbf{R}^2$ consisting of only two lines whose symmetry group is D_4. Write down all rotations and reflections preserving your subset. ***(4 marks)***

4 (i) State the Sylow theorems. You should carefully define all the terms and notation used. ***(5 marks)***

(ii) Let G be a group of order 91.

(a) Show that G has a normal subgroup N of order 13. ***(3 marks)***

(b) G also has a normal subgroup P of order 7 (no proof required). Prove that

$$pnp^{-1}n^{-1} = e \text{ for all } p \in P, n \in N.$$

(4 marks)

(c) Define a function $\phi : P \times N \to G$ by $\phi(x, y) = xy$. Prove that ϕ is an isomorphism of groups. ***(6 marks)***

(iii) (a) Give the definition of a simple group. ***(2 marks)***

(b) By considering the conjugation action of G on the set of Sylow 3-subgroups show that there is no simple group of order 36. ***(5 marks)***

End of Question Paper

Diagram for Question 2

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