The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

MAS422 Magnetohydrodynamics **2 hours**

Answer all four questions. Formulae are on the last page.

1 (i) Consider a purely toroidal magnetic field of the form

$$\mathbf{B} = B_0(r)\hat{\theta},$$

in a cylindrical coordinate system (r, θ, y). Here, $B_0(r)$ decreases with increasing r. Assume that the perfectly conducting plasma is initially at rest and is magnetically dominated by such a field, so that we can ignore gas pressure and force due to gravity. If the fluid's perturbed velocity vector is $\mathbf{v} = v_r\hat{\mathbf{r}} + v_y\hat{\mathbf{y}}$, show that the linearised induction equation for the perturbed magnetic field $\mathbf{B_1}$ is given by

$$\frac{\partial \mathbf{B_1}}{\partial t} = \frac{B_0}{r}\frac{\partial v_r}{\partial \theta}\hat{\mathbf{r}} + B_0\Phi\hat{\theta} + \frac{B_0}{r}\frac{\partial v_y}{\partial \theta}\hat{\mathbf{y}},$$

where $\Phi = -\left(\nabla\cdot\mathbf{v} - \dfrac{v_r}{r}\right) - \dfrac{v_r}{B_0(r)}\dfrac{\partial B_0(r)}{\partial r}$. ***(7 marks)***

Write down the equation for $\dfrac{\partial^2 \mathbf{v_1}}{\partial t^2}$, using the MHD momentum equation for such a magnetically dominated plasma in terms of Φ, using the definition for Alfven speed, v_A as $v_A(r) = B_0(r)/\sqrt{\mu_0\rho_0}$ where ρ_0 is the density and μ_0 is the permeability. ***(7 marks)***

(ii) Using a scalar potential $\psi(x, y)$, we can describe the magnetic field $\mathbf{B}$ as

$$\mathbf{B} = \nabla\psi \times \hat{\mathbf{z}}.$$

What partial differential equation in (x, y) coordinates does ψ satisfy? Find a non-trivial solution for this equation which has oscillatory behaviour in x. ***(11 marks)***

2 (i) Consider the magnetic induction equation in the case where the magnetic diffusivity $\eta = 0$.
Use $\nabla \cdot \mathbf{B} = 0$ and the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

to show that the induction equation may be written as

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{B}}{\rho}\right) + (\mathbf{v} \cdot \nabla)\frac{\mathbf{B}}{\rho} = \left(\frac{\mathbf{B}}{\rho} \cdot \nabla\right) \mathbf{v} \; .$$

(8 marks)

(ii) An inviscid, perfectly conducting, incompressible fluid, is permeated by a uniform magnetic field $\mathbf{B_0}$. The motion of the fluid is described by the momentum equation

$$\rho \left[\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{U}\right] = -\nabla p + \frac{(\nabla \times \mathbf{B})}{\mu_0} \times \mathbf{B}$$

and magnetic induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}).$$

The fluid is initially at rest and then given a small perturbation. Write down the linearised momentum and induction equations. ***(7 marks)***

(iii) Seeking solutions proportional to $\exp i(\mathbf{k} \cdot \mathbf{x} - \omega t)$ in the above linearised equations, show that

$$\omega^2 = \frac{(\mathbf{k} \cdot \mathbf{B_0})^2}{\mu_0 \rho_0}$$

where μ_0 is the magnetic permeability and ρ_0 the fluid density.

(10 marks)

3 (i) If a plasma is incompressible and the radius of a magnetic flux tube is decreased by a factor 3, use conservation of mass and flux to determine what happens to its length and field strength? ***(8 marks)***

(ii) The momentum equation can be written as

$$\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g}.$$

Suppose that $\mathbf{u} = 0$ and $\mathbf{g} = g\hat{z}$ where g is a constant. Show that the above equation can be written as an equation in z alone.
For $p = K\rho^{1+\frac{1}{n}}$ with both K and n constant, find ρ in terms of z.

(7 marks)

 Question 3 continued on next page

3 (continued)

(iii) If the magnetic field is given by

$$\mathbf{B} = -y\hat{\mathbf{x}} \ + \ \hat{\mathbf{y}},$$

calculate

(a) $\mathbf{J} \times \mathbf{B}$ ***(2 marks)***

(b) $(\mathbf{B} \cdot \nabla)\dfrac{\mathbf{B}}{\mu_0}$ ***(2 marks)***

(c) $-\nabla\left(\dfrac{B^2}{2\mu_0}\right)$ ***(2 marks)***

Sketch the field-lines denoting the directions with arrows and indicate clearly on your sketch the direction of the tension forces on $y = 0$.
(4 marks)

4 (i) State Ohm's law using electric field $\mathbf{E}$, fluid velocity $\mathbf{u}$, magnetic field $\mathbf{B}$ and current density $\mathbf{J}$. ***(2 marks)***

Using $\mathbf{u} = U_0(-x, y, 0)$, $\mathbf{B} = (0, B, 0)$ and zero electric field, show that

$$B(x) \propto \exp\left[-(U_0/2\eta)x^2\right],$$

where η is the magnetic diffusivity. ***(5 marks)***

(ii) For a linear force-free magnetic field,

$$\nabla \times \mathbf{B} = \alpha\mathbf{B},$$

where α is some function of position. What is the restriction on α and why? ***(3 marks)***

Consider a poloidal magnetic field of the form

$$\mathbf{B} = B_r\hat{r} + B_z\hat{z},$$

such that $\mathbf{B}$ is independent of azimuth ϕ.

Show that such an axisymmetric, force-free, poloidal magnetic field must be current free. ***(5 marks)***

 Question 4 continued on next page

4 (continued)

(iii) For a rotating object symmetric around a rotation axis, the velocity $\mathbf{v}$ in cylindrical coordinates (r, ϕ, z)

$$\mathbf{v} = r\Omega(r, z)\hat{\phi}$$

is independent of ϕ. Here, Ω is the angular velocity. Now, consider that the object has axisymmetric poloidal field, frozen into plasma. Show that a steady state is possible only if Ω is constant along field lines.
(Hint: use $\mathbf{B} = \nabla \times \frac{1}{r}\psi(r, z)\hat{\phi}$). ***(10 marks)***

 Question 4 continued on next page

4 (continued)

Formulae Sheet

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

	u	v	w	f	g	h
cartesian	x	y	z	1	1	1
spherical	r	θ	ϕ	1	r	$r\sin\theta$
cylindrical	r	ϕ	z	1	r	1

$$\nabla \cdot \mathbf{V} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghV_u) + \frac{\partial}{\partial v}(fhV_v) + \frac{\partial}{\partial w}(fgV_w)\right]$$

$$\nabla \times \mathbf{V} = \frac{1}{gh}\left[\frac{\partial}{\partial v}(hV_w) - \frac{\partial}{\partial w}(gV_v)\right]\hat{u} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(fV_u) - \frac{\partial}{\partial u}(hV_w)\right]\hat{v} + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gV_v) - \frac{\partial}{\partial v}(fV_u)\right]\hat{w}$$

vector identity:

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

End of Question Paper